Structural Analysis

Felix Udoeyo

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This book is dedicated to my wife, Dr. Joan Udoeyo, and to my children, Uduak, Ubong, and Idorenyin.

Contents

PART ONE
INTRODUCTION TO STRUCTURAL ANALYSIS AND STRUCTURAL LOADS

Chapter 1

Introduction to Structural Analysis

1.1 Structural Analysis Defined

A structure, as it relates to civil engineering, is a system of interconnected members used to support external loads. Structural analysis is the prediction of the response of structures to specified arbitrary external loads. During the preliminary structural design stage, a structure's potential external load is estimated, and the size of the structure's interconnected members are determined based on the estimated loads. Structural analysis establishes the relationship between a structural member's expected external load and the structure's corresponding developed internal stresses and displacements that occur within the member when in service. This is necessary to ensure that the structural members satisfy the safety and the serviceability requirements of the local building code and specifications of the area where the structure is located.

1.2 Types of Structures and Structural Members

There are several types of civil engineering structures, including buildings, bridges, towers, arches, and cables. Members or components that make up a structure can have different forms or shapes depending on their functional requirements. Structural members can be classified as beams, columns and tension structures, frames, and trusses. The features of these forms will be briefly discussed in this section.

1.2.1 Beams

Beams are structural members whose longitudinal dimensions are appreciably greater than their lateral dimensions. For example, the length of the beam, as shown in Figure 1.1, is significantly greater than its breadth and depth. The cross section of a beam can be rectangular, circular, or triangular, or it can be of what are referred to as standard sections, such as channels, tees, angles, and I-sections. Beams are always loaded in the longitudinal direction.

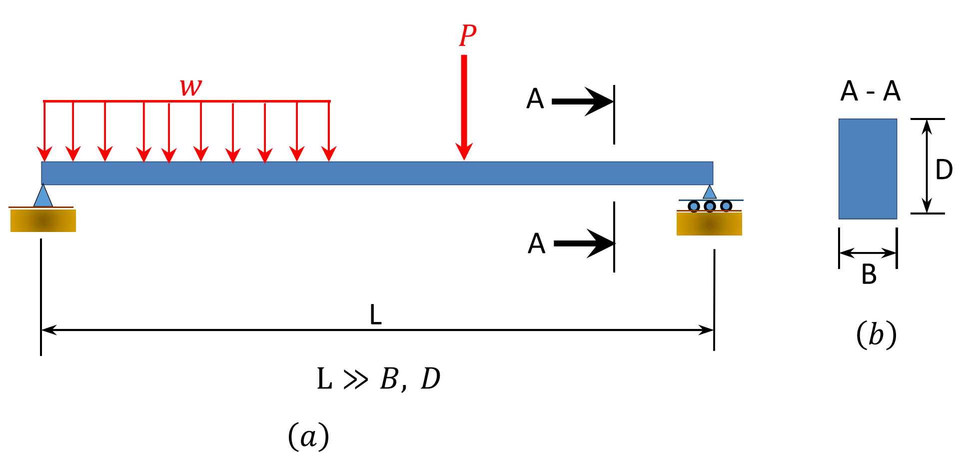
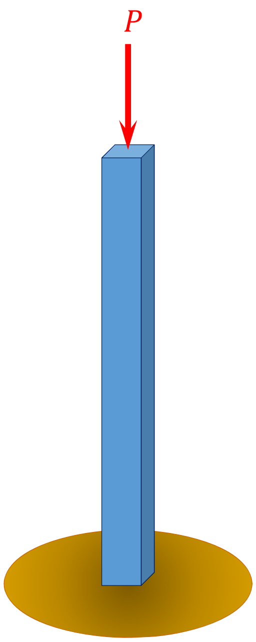


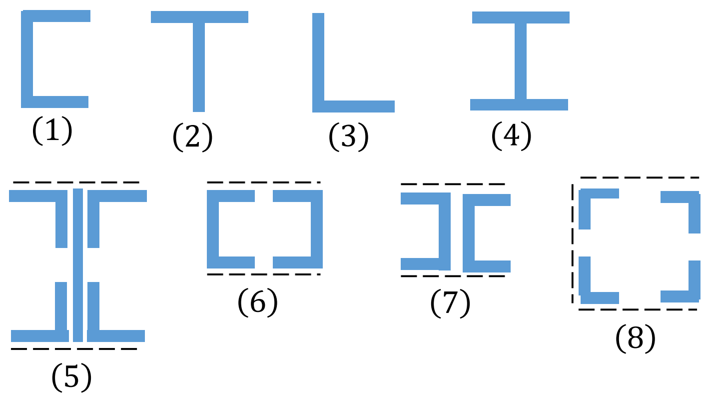
Fig. 1.1. Beam.

1.2.2 Columns and Tension Structures

Columns are vertical structural members that are subjected to axial compression, as shown in figure 1.2a. They are also referred to as struts or stanchions. Columns can be circular, square, or rectangular in their cross sections, and they can also be of standard sections. In some engineering applications, where a single-member strength may not be adequate to sustain a given load, built-up columns are used. A built-up column is composed of two or more standard sections, as shown in Figure 1.2b. Tension structures are similar to columns, with the exception that they are subjected to axial tension.



(a)



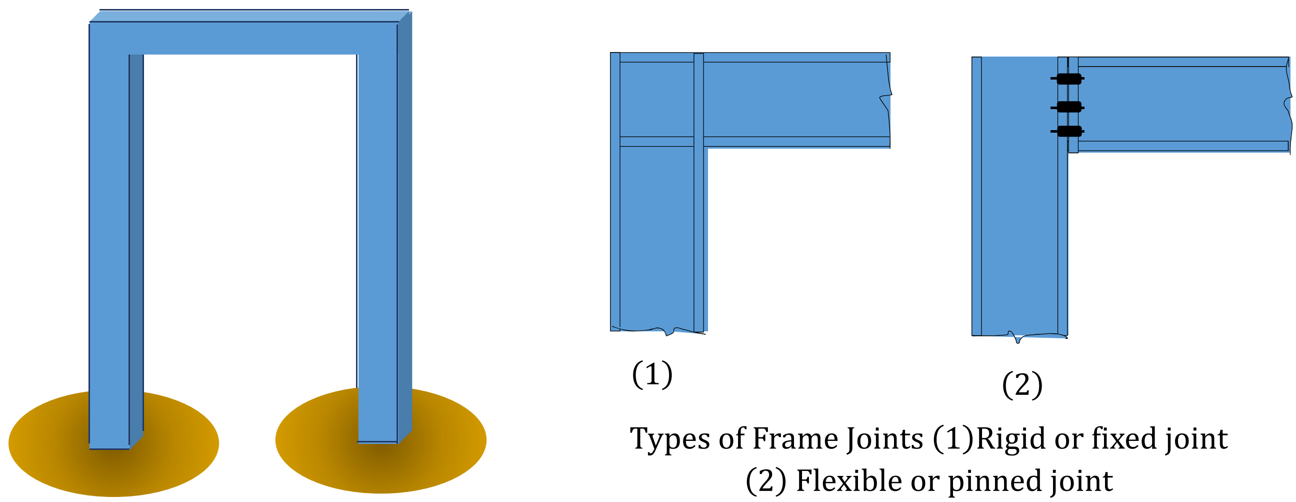
Standard sections: (1) channel, (2) tee, (3) angle, and (4) wide flange beam; Built-up sections: (5) four angles and a plate, (6) two channels (front-to-front), (7) two channels (back-to-back), and (8) four angles (front-to-front).

(b)

Fig. 1.2. Columns.

1.2.3 Frames

Frames are structures composed of vertical and horizontal members, as shown in Figure 1.3a. The vertical members are called columns, and the horizontal members are called beams. Frames are classified as sway or non-sway. A sway frame allows a lateral or sideward movement, while a non-sway frame does not allow movement in the horizontal direction. The lateral movement of the sway frames are accounted for in their analysis. Frames can also be classified as rigid or flexible. The joints of a rigid frame are fixed, whereas those of a flexible frame are moveable, as shown in Figure 1.3b.



(a) (b)

Fig. 1.3. Frame.

1.2.4 Trusses

Trusses are structural frameworks composed of straight members connected at the joints, as shown in Figure 1.4. In the analysis of trusses, loads are applied at the joints, and members are assumed to be connected at the joints using frictionless pins.

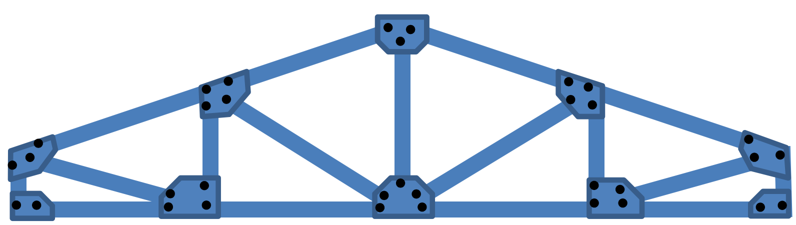


Fig. 1.4. Truss.

1.3 Fundamental Concepts and Principles of Structural Analysis

1.3.1 Equilibrium Conditions

Civil engineering structures are designed to be at rest when acted upon by external forces. A structure at rest must satisfy the equilibrium conditions, which require that the resultant force and the resultant moment acting on a structure be equal to zero. The equilibrium conditions of a structure can be expressed mathematically as follows:

$$\sum F = 0, \text{ and } \sum M = 0 \tag{1.1}$$

1.3.2 Compatibility of Displacement

The compatibility of displacement concept implies that when a structure deforms, members of the structure that are connected at a point remain connected at that point without void or hole. In other words, two parts of a structure are said to be compatible in displacements if the parts remain fitted together when the structure deforms due to the applied load. Compatibility of displacement is a powerful concept used in the analysis of indeterminate structures with unknown redundant forces in excess of the three equations of equilibrium. For an illustration of the concept, consider the propped cantilever beam shown in Figure 1.5a. There are four unknown reactions in the beam: the reactive moment, a vertical and horizontal reaction at the fixed end, and another vertical reaction at the prop at point *B*. To determine the unknown reactions in the beam, one more equation must be added to the three equations of equilibrium. The additional equation can be obtained as follows, considering the compatibility of the structure:

$$\Delta_{BP} + \Delta_{BR} = 0 \tag{1.2}$$

In this equation, Δ_{BP} is the displacement at point *B* of the structure due to the applied load *P* (Figure 1.5b), and Δ_{BR} is the displacement at point *B* due to the reaction at the prop *R* (Figure 1.5c). Students should always remember that the first subscript of the displacement indicates the location where the displacement occurs, while the second subscript indicates the load causing the displacement.

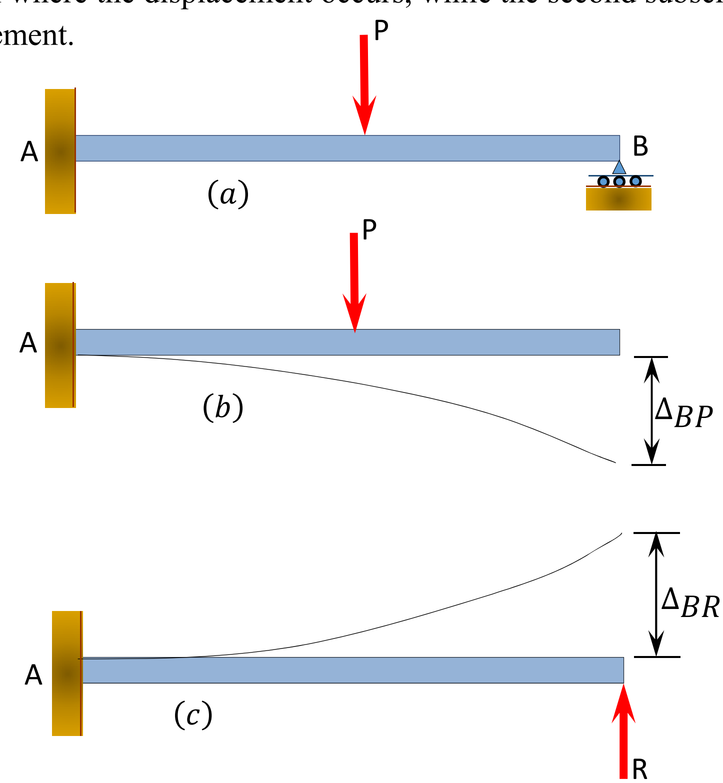


Fig. 1.5. Propped cantilever beam.

1.3.3 Principle of Superposition

The principle of superposition is another very important principle used in structural analysis. The principle states that the load effects caused by two or more loadings in a linearly elastic structure are equal to the sum of the load effects caused by the individual loading. For an illustration, consider the cantilever beam carrying two concentrated loads P_1, and P_2, in Figure 1.6a. Figures 1.6b and 1.6c are the responses of the structure in terms of the displacement at the free end of the beam when acted upon by the individual loads. By the principle of superposition, the displacement at the free end of the beam is the algebraic sum of the displacements caused by the individual loads. This can is written as follows:

$$\Delta_B = \Delta_{BP_1} + \Delta_{BP_2} \tag{1.3}$$

In this equation, Δ_B is the displacement at B; Δ_{BP_1} and Δ_{BP_2} are the displacements at B caused by the loads P_1 and P_2, respectively.

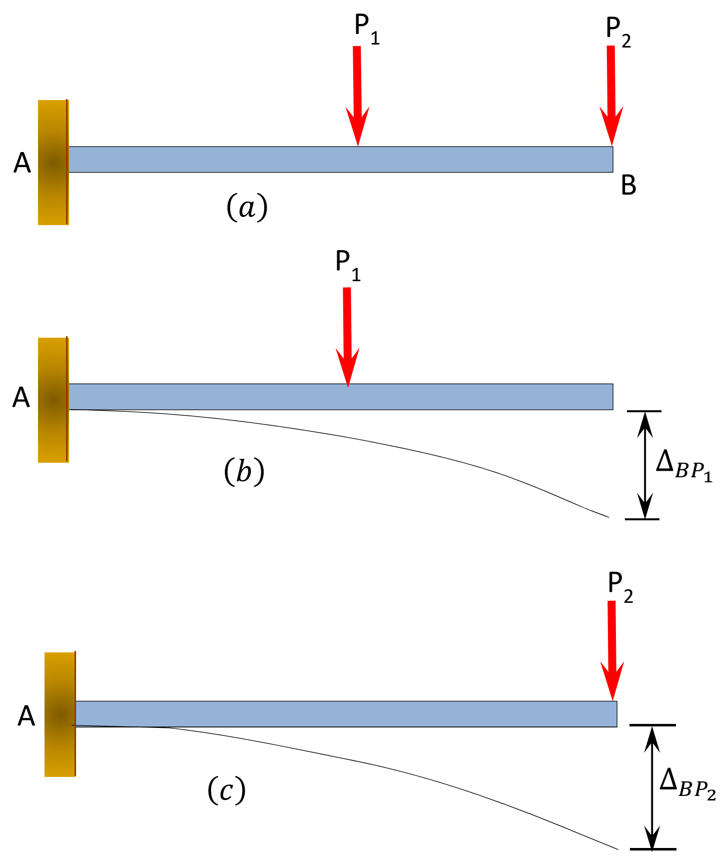


Fig. 1.6. Application of the principle of superposition.

1.3.4 Work-Energy Principle

The work-energy principle is a very powerful tool in structural analysis. Work is defined as the product of the force and the distance traveled by the force, while energy is defined as the ability to do work. Work can be transformed into various energy, including kinetic energy, potential energy, and strain energy. In the case of a structural system, based on the law of conservation of energy, work done W is equal to the strain energy U stored when deforming the system. This is expressed mathematically as follows:

$$W = U \tag{1.4}$$

Consider a case where a force F is gradually applied to a deformable structural system. By plotting the applied force against the deformation Δ of the structure, the load-deformation plot shown in Figure 1.7a is created. In the case of linearly elastic structure, the load-deformation diagram will be as shown in Figure 1.7b. The incremental work done dW by the force when deforming the structure over an incremental displacement $d\Delta$ is expressed as follows:

$$dW = Fd\Delta \tag{1.5}$$

The total work done is represented as follows:

$$W = \int_0^\Delta dW = \int_0^\Delta Fd\Delta \tag{1.6}$$

Thus, the strain energy is written as follows:

$$U = \int_0^\Delta Fd\Delta \tag{1.7}$$

The strain energy in the case of linearly elastic deformation can be obtained by computing the area under the load-deformation diagram in Figure 1.7b. This is expressed as follows:

$$U = \tfrac{1}{2}F\Delta \tag{1.8}$$

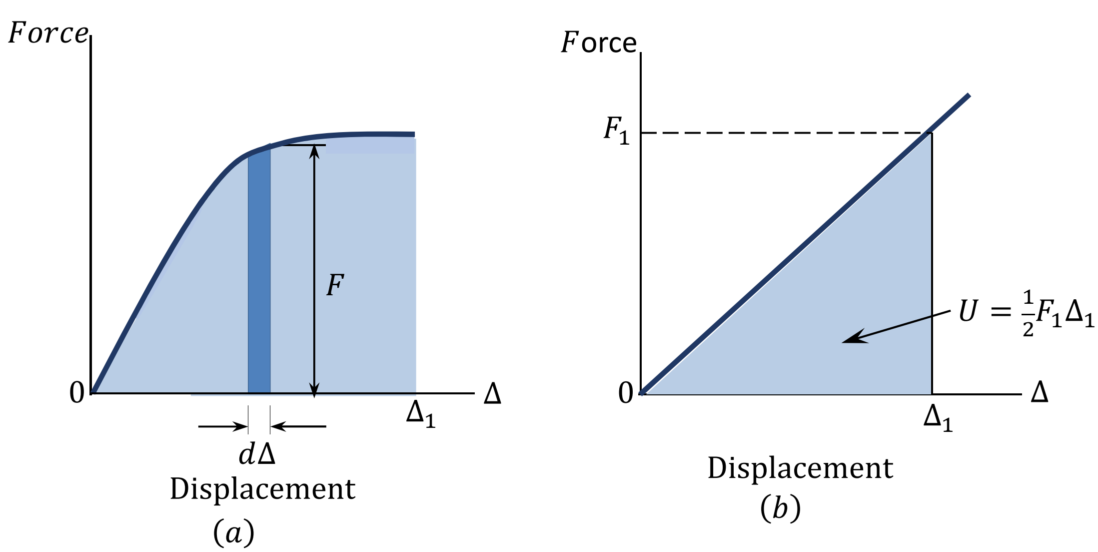


Fig. 1.7. Load-deformation diagram.

1.3.5 Virtual Work Principle

The virtual work principle is another powerful and useful analytical tool in structural analysis. It was developed in 1717 by Johann Bernoulli. Virtual work is defined as the work done by a virtual or imaginary force acting on a deformable body through a real distance, or the work done by a real force acting on a rigid body through a virtual or fictitious displacement. To formulate this principle in the case of virtual displacements through a rigid body, consider a propped cantilever beam subjected to a concentrated load P at a distance x from the fixed end, as shown in Figure 1.8a. Suppose the beam undergoes an elementary virtual displacement δu at the propped end, as shown in Figure 1.8b. The total virtual work performed is expressed as follows:

$$\delta W \ = R_B \delta_u - P\frac{x}{L}\delta u \tag{1.9}$$

Since the beam is in equilibrium, $\delta W = 0$ (by the definition of the principle of virtual work of a body).

The principle of virtual work of a rigid body states that if a rigid body is in equilibrium, the total virtual work performed by all the external forces acting on the body is zero for any virtual displacement.

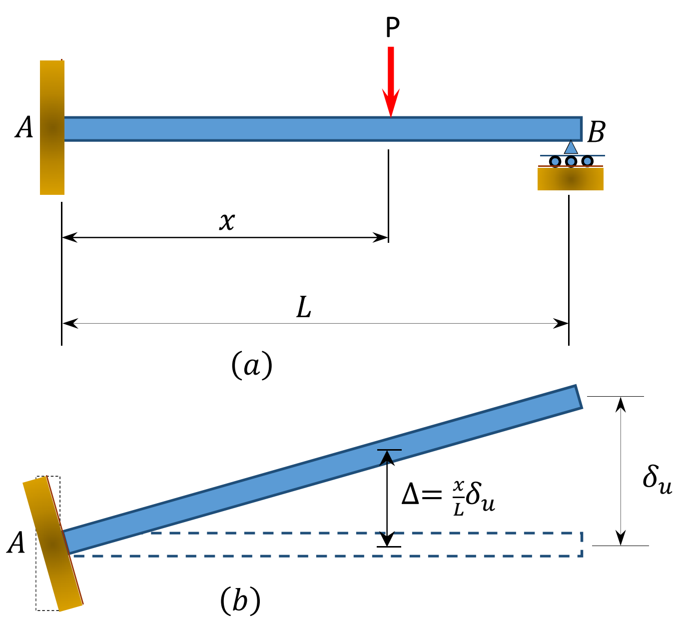


Fig. 1.8. Propped cantilever beam.

1.3.6 Structural Idealization

Structural idealization is a process in which an actual structure and the loads acting on it are replaced by simpler models for the purpose of analysis. Civil engineering structures and their loads are most often complex and thus require rigorous analysis. To make analysis less cumbersome, structures are represented in simplified forms. The choice of an appropriate simplified model is a very important aspect of the analysis process, since the predictive response of such idealization must be the same as that of the actual structure. Figure 1.9a shows a simply supported wide-flange beam structure and its load. The plan of the same beam is shown in Figure 1.9b, and the idealization of the beam is shown in Figure 1.9c. In the idealized form, the beam is represented as a line along the beam's neutral axis, and the load acting on the beam is shown as a point or concentrated load because the load occupies an area that is significantly less than the total area of the structure's surface in the plane of its application. Figures 1.10a and 1.10b depict a frame and its idealization, respectively. In the idealized form, the two columns and the beam of the frame are represented by lines passing through their respective neutral axes. Figures 1.11a and 1.11b show a truss and its idealization. Members of the truss are represented by lines passing through their respective neutral axes, and the connection of members at the joints are assumed to be by frictionless pins.

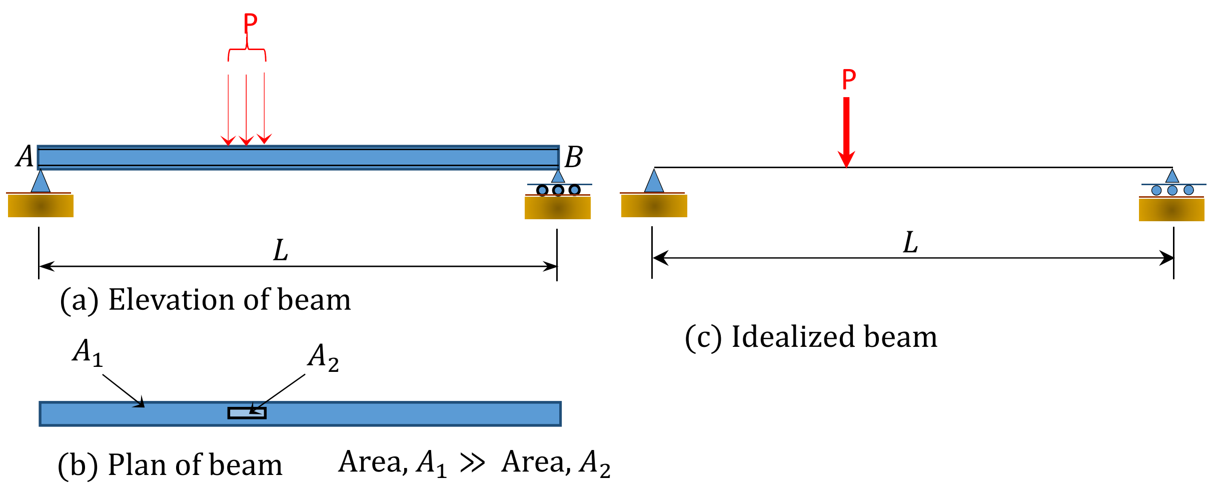


Fig. 1.9. Wide – flange beam idealization.

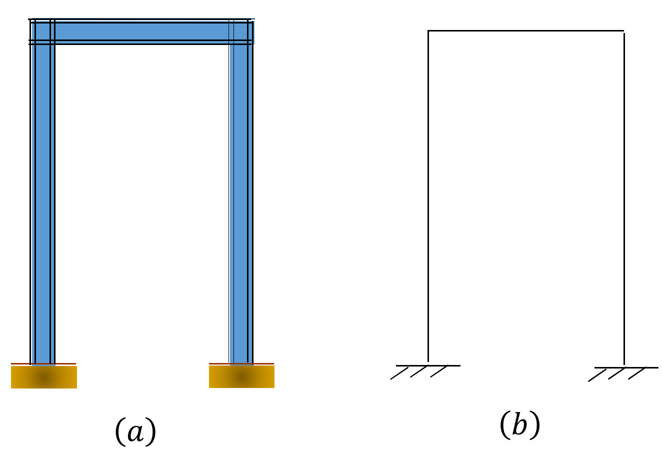


Fig. 1.10. Frame idealization.

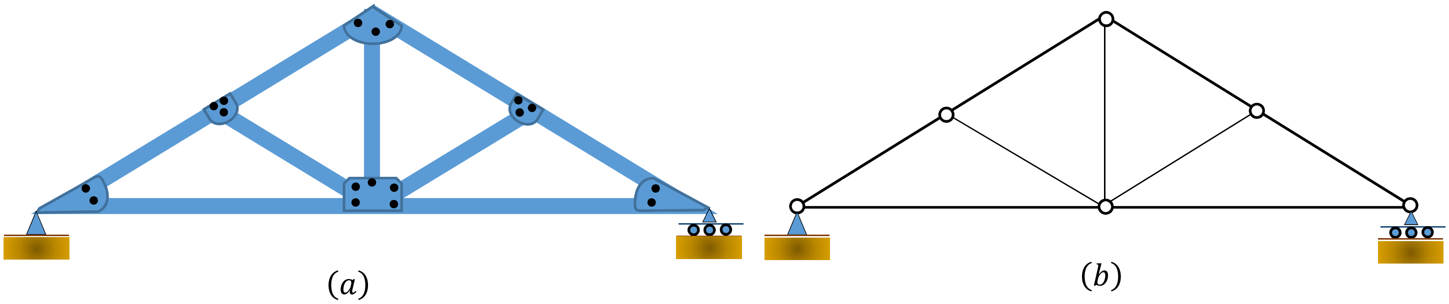


Fig. 1.11. Truss idealization.

1.3.7 Method of Sections

The method of sections is useful when determining the internal forces in structural members that are in equilibrium. The method involves passing an imaginary section through the structural member so that it divides the structure into two parts. Member forces are determined by considering the equilibrium of either part. For a beam in equilibrium that is subjected to transverse loading, as shown in Figure 1.12, the internal forces include an axial or normal force, N, shear force, V, and bending moments, M.

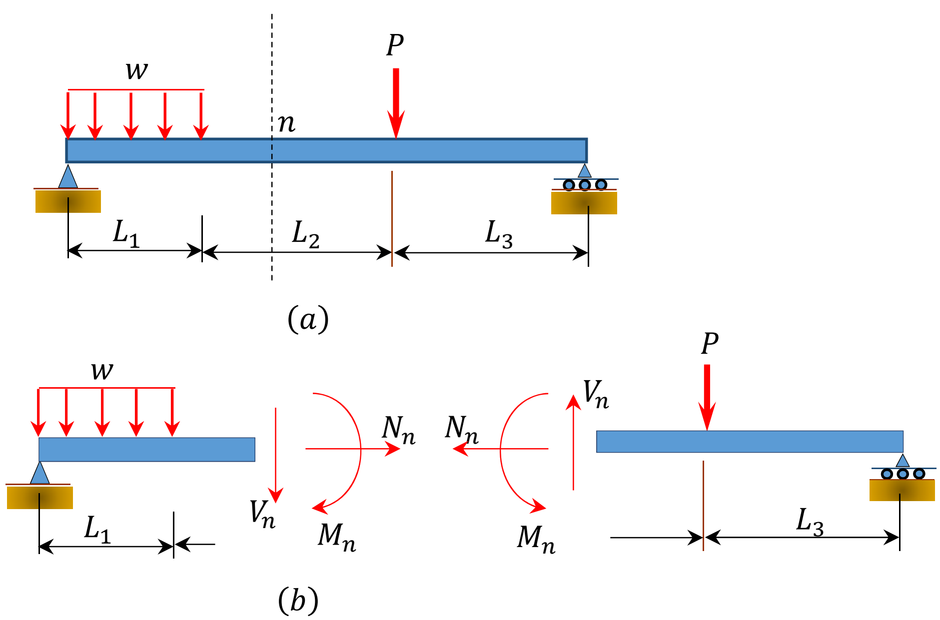


Fig. 1.12. Beam in equilibrium subjected to transverse loading.

1.3.8 Free-Body Diagram

A free-body diagram is a diagram showing all the forces and moments acting on the whole or a portion of a structure. A free-body diagram must also be in equilibrium with the actual structure.

The free-body diagram of the entire beam shown in Figure 1.13a is depicted in Figure 1.13b. If the free-body diagram of a segment of the beam is desired, the segment will be isolated from the entire beam using the method of sections. Then, all the external forces on the segment and the internal forces from the adjoining part of the structure will be applied to the isolated part.

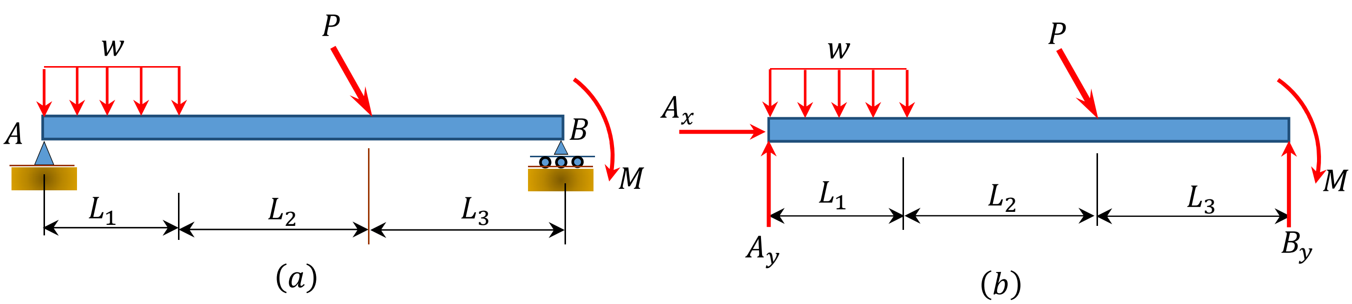


Fig. 1.13 Freebody diagram of a beam.

1.4 Units of Measurement

The two most commonly used systems in science and technology are the International System of Units (SI Units) and the United States Customary System (USCS).

1.4.1 International System of Units

In the SI units, the arbitrarily defined base units include meter (m) for length, kilogram (kg) for mass, and second (s) for time The unit of force, newton (N), is derived from Newton's second law. One newton is the force required to give a kilogram of mass an acceleration of 1 m/s^2. The magnitude, in newton, of the weight of a body of mass *m* is written as follows:

$$W\ (N) = m\ (kg) \times g\ (m/s^2)$$

where

$g = 9.81$ m/s^2

1.4.2 United States Customary System

In the United States Customary System, the base units include foot (ft) for length, second (s) for time, and pound (lb) for force. The slug for mass is a derived unit. One slug is the mass accelerated at 1 ft/s^2 by a force of 1 lb. The mass of a body, in slug, is determined as follows:

$$m\ (slugs) = \frac{W\ (lb)}{g\ (\frac{ft}{s^2})}, \text{ where } g = 32.2\ ft/s^2$$

The two systems of units are summarized in Table 1.1 below.

Table 1.1. Comparison of unit measurement systems.

Quantity	Length	Time	Mass	Force
Dimensional Symbol	L	T	M	F
U.S. Customary Units	foot (ft)	second (s)	Slug	pound (lb)
SI Units	meter (m)	second (s)	kilogram (kg)	Newton (N)

Table 1.2. Unit conversion.

Quantity	U.S. Customary Unit	Equal	SI Unit
Acceleration	ft/s^2		$0.3048\ m/s^2$
Area	in^2		$645.2\ mm^2$
Density	lb/ft^3		$16.02\ kg/m^3$
Energy, Work	in.lb		0.113 N.m (Joule, J)
Force	lb		4.448 N
	kip		4.448 kN
Impulse	lb.s		4.448 N.s
Length	in		25.4 mm
	ft		0.3048 m
Mass	Slug		14.59 kg
Moment of a couple	lb.in		0.113 N.m
	k.ft		1356 N.m
Moment of inertia of area	in^4		$0.4162 \times 10^{-6}\ m^4$
	ft^4		$8.6303\ x\ 10^{-3}\ m^4$
Moment of inertia of mass	$lb.ft.s^2$		$1.356\ kg.m^2$
Momentum	lb.s		4.448 kg.m/s
Power	ft.lb/s		1.356 W
Pressure	psi		6.895 kPa
	ksi		6.895 MPa
Velocity	ft/s		0.3048 m/s
Volume of an object	ft^3		$0.02832\ m^3$
Volume of a liquid	gal		3.785 L

1.4.3 SI Prefixes

Prefixes are used in the International System of Units when numerical quantities are quite large or small. Some of these prefixes are presented in Table 1.3.

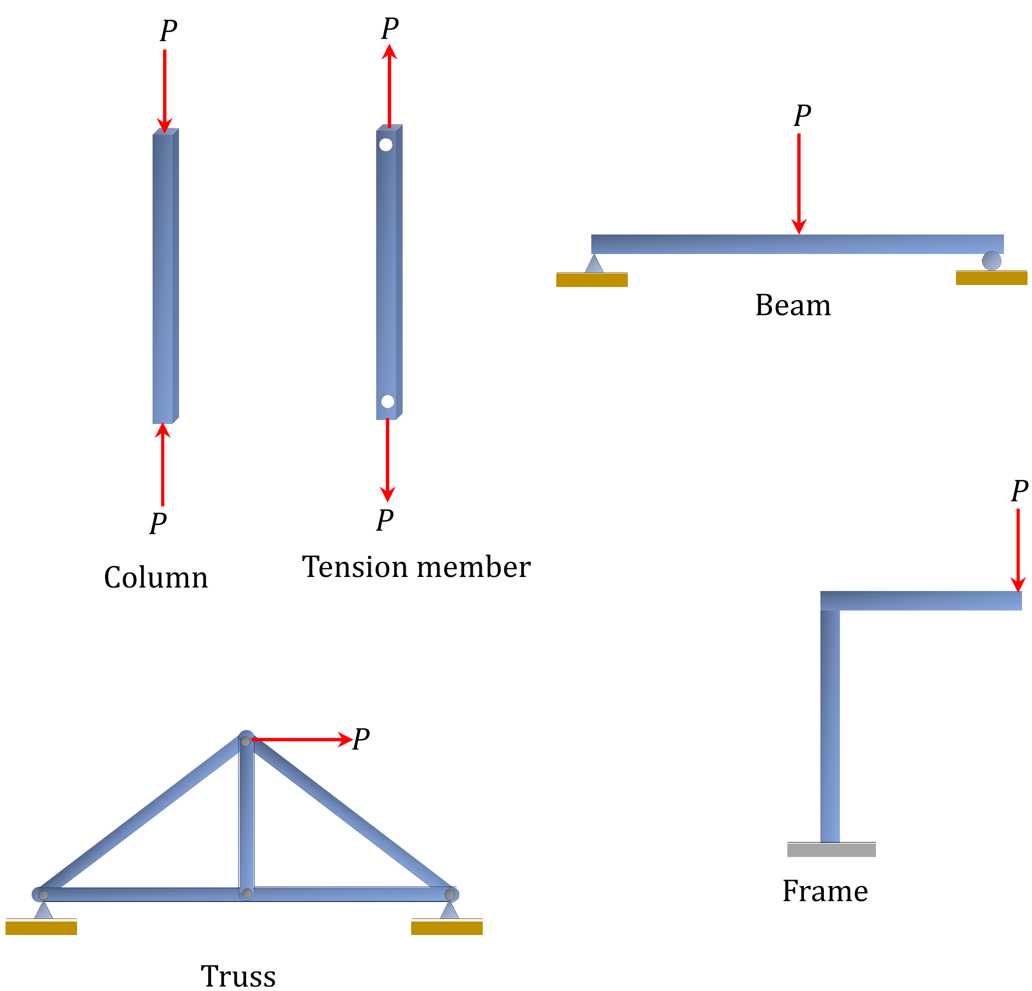
Table 1.3. SI prefixes.

Multiplication Factor	Exponential Form	Prefix	Symbol
1 000 000 000 000	10^{12}	Tera	T
1 000 000 000	10^{9}	Giga	G
1 000 000	10^{6}	Mega	M
1 000	10^{3}	Kilo	K
0.001	10^{-3}	Milli	M
0.000 001	10^{-6}	Micro	M
0.000 000 001	10^{-9}	Nano	N

Chapter Summary

Introduction to structural analysis: Structural analysis is defined as the prediction of structures' behavior when subjected to specified arbitrary external loads.

Types of structures: Structural members can be classified as beams, columns and tension structures, frames, and trusses.



Fundamental concepts of structural analysis: The fundamental concept and principles of structural analysis discussed in the chapter include equilibrium conditions, compatibility of displacement, principle of superposition, work-energy principle, virtual work principle, structural idealization, method of sections, and free-body diagram.

Chapter 2

Structural Loads and Loading System

2.1 Types of Structural Loads

Civil engineering structures are designed to sustain various types of loads and possible combinations of loads that could act on them during their lifetime. Accurate estimation of the magnitudes of these loads is a very important aspect of the structural analysis process. There are local and international codes, as well as research reports and documents, that aid designers in this regard. Structural loads can be broadly classified into four groups: dead loads, live loads, impact loads, and environmental loads. These loads are briefly described in the following sections.

2.1.1 Dead Loads

Dead loads are structural loads of a constant magnitude over time. They include the self-weight of structural members, such as walls, plasters, ceilings, floors, beams, columns, and roofs. Dead loads also include the loads of fixtures that are permanently attached to the structure. Prior to the analysis and design of structures, members are preliminarily sized based on architectural drawings and other relevant documents, and their weights are determined using the information available in most codes and other civil engineering literature. The recommended weight values of some commonly used materials for structural members are presented in Table 2.1. The determination of the dead load due to structural members is an iterative process. During design, member sizes and weight could change, and the process is repeated until a final member size is obtained that could support the member's weight and the superimposed loads.

Table 2.1. Unit weights of construction materials.

Material	Unit Weight	
	lb/ft^3	kN/m^3
Reinforced concrete	150	23.60
Plain concrete	145	22.60
Structural steel	490	77.00
Aluminum	165	25.90
Brick	120	18.90
Concrete masonry unit	135	21.20
Wood (Douglas fir larch)	34	5.30
Engineered wood (plywood)	36	5.7

Example 2.1

The semi-gravity retaining wall shown in Figure 2.1 Figure 2.1 is made of mass concrete with a unit weight of $23.6\ \mathrm{kN/m^3}$. Determine the length of the wall's weight per foot.

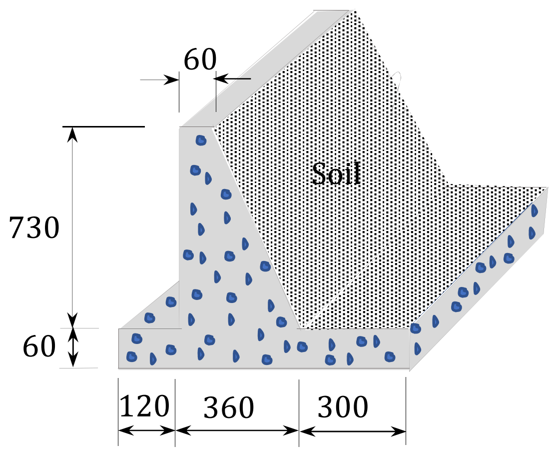


Fig. 2.1. Semi − gravity retaining wall (all dimensions in cm).

Solution

Area of wall $= (7.8\ \mathrm{m})(0.6\ \mathrm{m}) + (7.3\ \mathrm{m})(0.6\ \mathrm{m}) + \left(\frac{1}{2}\right)(3\ \mathrm{m})(7.3\ \mathrm{m}) = 20.01\ \mathrm{m^2}$

Length of the wall's weight per foot $= 20.01\ \mathrm{m^2} \times (23.6\ \mathrm{kN/m^3}) = 472.24\ \mathrm{kN/m}$

2.1.2 Live Loads

Live loads are moveable or temporarily attached to a structure. They include the loads on a building created by the storage of furniture and equipment, occupancy (people), and impact. Typical live load values are presented in Table 2.2. The loads were obtained from Table 4.3-1 in ASCE 7-16.

Table 2.2. Minimum uniform and concentrated floor live loads.

Occupancy or Use	Live Load	
	Uniform psf (kN/m^2)	Concentrated lb (kN)
Residential dwellings, apartments, hotels		
Private rooms and corridors serving them	40 (1.92)	
Public rooms and corridors serving them	100 (4.79)	
Hospitals		
Patient rooms	40 (1.92)	1,000 (4.45)
Operating rooms, laboratories	60 (2.87)	1,000 (4.45)
Corridors above first floor	80 (3.83)	1,000 (4.45)
Office buildings		
Lobbies and first floor corridors	100 (4.79)	2,000 (8.90)
Offices	50 (2.40)	2,000 (8.90)
Corridors above first floor	80 (3.83)	2,000 (8.90)
Recreational uses		
Bowling alleys, poolrooms, and similar uses	75 (3.59)	
Dance halls and ballrooms, gymnasiums	100 (4.79)	
Stadiums and arenas with fixed seats	60 (2.87)	
Stores		
Retail		
First floor	100 (4.79)	1,000 (4.45)
Upper floors	75 (3.59)	1,000 (4.45)
Wholesale, all floors	125 (6.00)	1,000 (4.45)
Storage warehouses		
Light	125 (6.00)	
Heavy	250 (11.97)	
Manufacturing		
Light	125 (6.00)	2,000 (8.90)
Heavy	250 (11.97)	3,000 (13.40)
Schools		
Classrooms	40 (1.92)	1,000 (4.45)
Corridors above first floor	80 (3.83)	1,000 (4.45)
First floor corridors	100 (4.79)	1,000 (4.45)

Example 2.2

The floor system of the classroom shown in Figure 2.2 consists of a 3-inch-thick reinforced concrete slab supported by steel beams. If the weight of each steel beam is 62 lb/ft, determine the dead load in lb/ft supported by any interior beam.

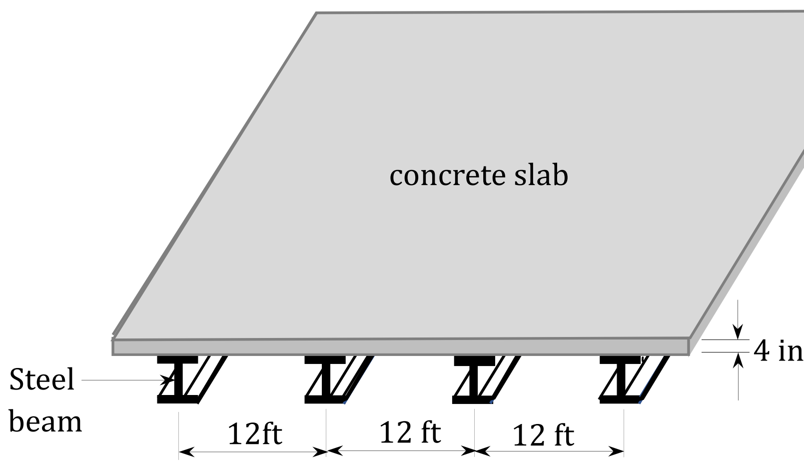


Fig. 2.2. Classroom floor system.

Solution

Dead load due to slab weight $= (12\text{ft})\left(\frac{4\text{in}}{12}\right)(150\ \text{lb/ft}^3) = 600\ \text{lb/ft}$
Dead load due to beam weight $= 62\ \text{lb/ft}$
Live load due to occupancy or use (classroom) $= (40\ \text{lb/ft}^2)(12\ \text{ft}) = 480\ \text{lb/ft}$
Total uniform load on steel beam $= 1142\ \text{lb/ft} = 1.142\ \text{k/ft}$

2.1.3 Impact Loads

Impact loads are sudden or rapid loads applied on a structure over a relatively short period of time compared with other structural loads. They cause larger stresses in structural members than those produced by gradually applied loads of the same magnitude. Examples of impact loads are loads from moving vehicles, vibrating machinery, or dropped weights. In practice, impact loads are considered equal to imposed loads that are incremented by some percentage, called the impact factor. Some building load impact factors are presented in Table 2.3. The American Association of State Highway and Transportation Officials (AASHTO) specifies the following expression for the computation of the impact factor for a moving truck load for use in highway bridge design:

$$I = \frac{50}{L+125} \leq 0.3 \qquad \text{U.S. customary units}$$

$$I = \frac{15.2}{L+38.1} \leq 0.3 \qquad \text{SI units}$$

where

I = impact factor.
L = length in feet (or meters) of the span-loaded segment to cause maximum stress in the member under consideration.

Table 2.3. Building live load impact factors, as specified in *ASCE/SEI 7-16*.

Loading Case	I(%)
Elevator supports and machinery	100
Light machinery supports	20
Reciprocating machine supports	50
Hangers supporting floors and balconies	33
Crane support girders and their connections	25

2.1.4 Environmental Loads

2.1.4.1 Rain Loads

Rain loads are loads due to the accumulated mass of water on a rooftop during a rainstorm or major precipitation. This process, which is referred to as ponding, mostly occurs in flat roofs and roofs with pitches of less than 0.25 in/feet. Ponding in roofs occurs when the run off after precipitation is less than the amount of water retained on the roof. Water accumulated on a flat or low-pitch roof during a rainstorm can create a major structural load. Therefore, it must be considered when designing a building. The International Code Council requires that roofs with parapets include primary and secondary drains. The primary drain collects water from the roof and directs it to the sewer, while the secondary drain serves as a backup in the event that the primary drain is clogged. Figure 2.3 depicts a roof and these drainage systems. Section 8.3 of ASCE7-16 specifies the following equation for the computation of rain loads on an undeflected roof in the event that the primary drain is blocked:

$$R = 5.2\,(d_s + d_h) \quad \text{U.S. customary unit}$$
$$R = 0.0098\,(d_s + d_h) \quad \text{SI units}$$

where

R = rain load on the undeflected roof, in psi or KN/m^2.
d_s = depth of water on the undeflected roof up to the inlet of the secondary drainage system (i.e. the static head), in inches or mm.
d_h = additional depth of water on the undeflected roof above the inlet of the secondary drainage system (i.e. the hydraulic head), in inches or mm. It depends on the flow rate, the size of the drainage, and the area drained by each drain.

The flow rate, Q, in gallons per minute, can be computed as follows:

$$Q\ (\text{gpm}) = 0.0104\,Ai$$

where

A = roof area in square feet drained by the drainage system.
i = 100-yr., 1-hr. rainfall intensity in inches per hour for the building location specified in the plumbing code.

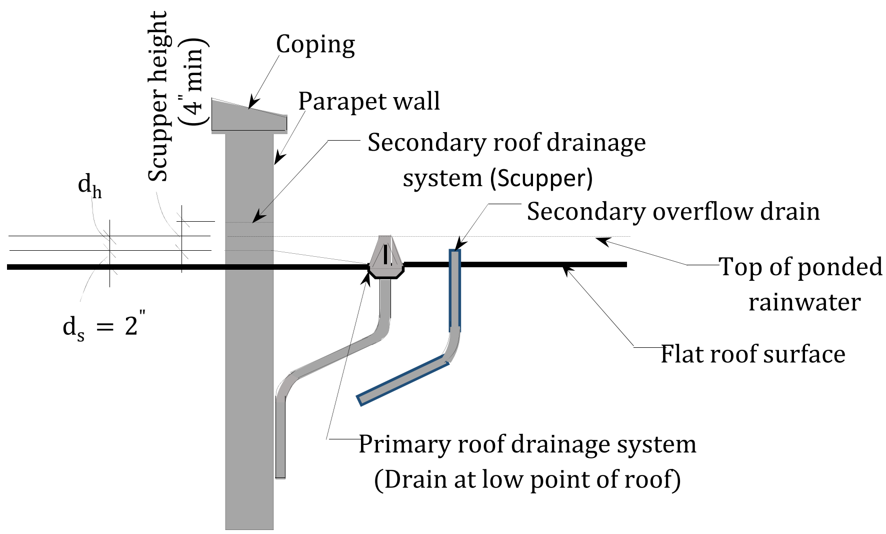


Fig. 2.3. Roof drainage system (Adapted from the International Code Council).

2.1.4.2 Wind Loads

Wind loads are pressures exacted on structures by wind flow. Wind forces have been the cause of many structural failures in history, especially in coastal regions. The speed and direction of wind flow varies continuously, making it difficult to predict the exact pressure applied by wind on existing structures. This explains the reason for the considerable research efforts on the effect and estimation of wind forces. Figure 2.4 shows a typical wind load distribution on a structure. Based on Bernoulli's principle, the relationship between dynamic wind pressure and wind velocity can be expressed as follows when visualizing the flow of wind as that of a fluid:

$$q = \frac{1}{2}\rho V^2 \tag{2.1}$$

where

q = dynamic wind pressure air in pounds per square foot.
ρ = mass density of air.
V = wind velocity in miles per hour.

Basic wind speed for specific locations in the continental United States can be obtained from the basic speed contour map in *ASCE 7-16*.

Assuming that the unit weight of air for a standard atmosphere is 0.07651 lb/ft^3 and substituting this value into the previously stated equation 2.1, the following equation can be used for static wind pressure:

$$q = \left(\frac{0.0765}{32.2}\right)\left(\frac{5280}{3600}\right)^2 \frac{V^2}{2} = 0.00256V^2 \tag{2.2}$$

To determine the magnitude of wind velocity and its pressure at various elevations above ground level, the *ASCE 7-16* modified equation 2.2 by introducing some factors to account for the height of the structure above ground level, the importance of the structure in regard to human life and property, and the topography of its location, as follows:

$$q_z = 0.00256K_zK_{zt}K_dK_eV^2 \quad \text{Customary units (lb/ft}^2\text{)}$$

$$q_z = 0.613K_zK_{zt}K_dK_eV^2 \quad \text{SI units (N/m}^2\text{)} \tag{2.3}$$

where

K_z = the velocity pressure coefficient that depends on the height of the structure and the exposure condition. The values of K_z are listed in Table 2.4.
K_{zt} = a topographic factor that accounts for an increase in wind velocity due to sudden changes in topography where there are hills and escarpments. This factor is an equal unity for building on level ground and increases with elevation.
K_d = wind directionality factor. It accounts for the reduced probability of maximum wind coming from any given direction and for the reduced probability of the maximum pressure developing on any wind direction most unfavorable to the structure. For structures subjected to wind loads only, $K_d = 1$; for structures subjected to other loads, in addition to a wind load, K_d values are tabulated in Table 2.5.
K_e = ground elevation factor. According to section 26.9 in *ASCE 7-16*, it is expressed as $K_e = 1$ for all elevations.
V = velocity of wind measured at a height z above ground level.

The three exposure conditions categorized as B, C, and D in Table 2.4 are defined in terms of surface roughness, as follows:

Exposure B: The surface roughness for this category includes urban and suburban areas, wooden areas, or other terrain with closely spaced obstructions. This category applies to buildings with mean roof heights of ≤ 30 ft (9.1 m) if the surface extends in the upwind direction for a distance greater than 1,500 ft. For buildings with mean roof heights greater than 30 ft (9.1 m), this category will apply if the surface roughness in the upwind direction is greater than 2,600 ft (792 m) or 20 times the height of the building, whichever is greater.

Exposure C: Exposure C applies where surface roughness C prevails. Surface roughness C includes open terrain with scattered obstructions having heights less than 30 ft.

Exposure D: The surface roughness for this category includes flats, smooth mud flats, salt flats, unbroken ice, unobstructed areas, and water surfaces. Exposure D applies where surface roughness D extends in the upwind direction for a distance greater than 5,000 ft or 20 times the building height, whichever is greater. This also applies if the surface roughness upwind is B or C, and the site is within 600 ft (183 m) or 20 times the building height, whichever is greater.

Table 2.4. Velocity pressure exposure coefficient, K_z, as specified in *ASCE 7-16*.

Height z above ground level ft (m)	K_Z		
	Exposure		
	B	C	D
0-15 (0-4.6)	0.57 (0.70) *	0.85	1.03
20 (6.1)	0.62 (0.70)	0.90	1.08
25 (7.6)	0.66 (0.70)	0.94	1.12
30 (9.1)	0.70	0.98	1.16
40 (12.2)	0.76	1.04	1.22
50 (15.2)	0.81	1.09	1.27
60 (18.0)	0.85	1.13	1.31
70 (21.3)	0.89	1.17	1.34
80 (24.4)	0.93	1.21	1.38
90 (27.4)	0.96	1.24	1.48

Table 2.5. Wind directional factor, K_d, as specified in *ASCE 7-16*.

Structure Type	K_d
Main wind force resisting system (MWFRS) Components and cladding	0.85 0.85
Arched roofs	0.85
Chimneys, tanks, and similar structures Square Hexagonal Round	 0.9 0.95 0.95
Solid freestanding walls and solid freestanding and attached signs	0.85
Open signs and lattice framework	0.85
Trussed towers Triangular, square, rectangular All other cross sections	 0.85 0.95

To obtain the final external pressures for the design of structures, equation 2.3 is further modified, as follows:

$$P_z = q_z G C_p \tag{2.4}$$

where

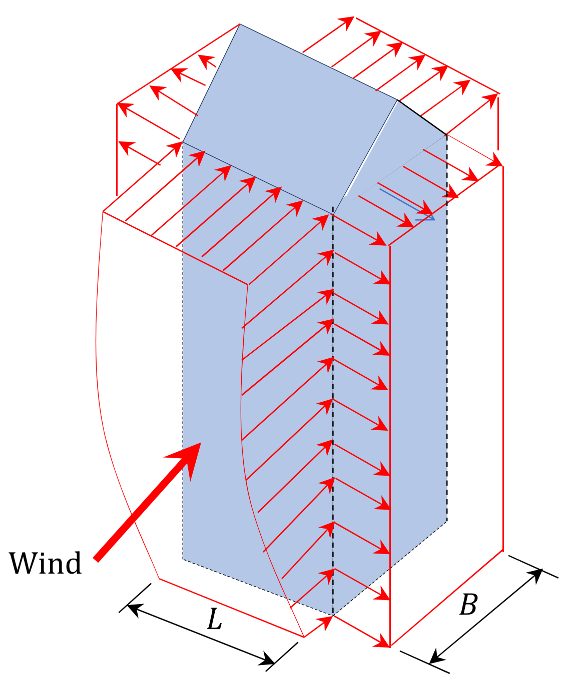
P_z = design wind pressure on a face of the structure at height z above ground level. It increases with the height on the windward wall, but it is constant with the height on the leeward and side walls.
G = gust effect factor. $G = 0.85$ for rigid structures with a natural frequency of ≥ 1 Hz. The gust factors for flexible structures are calculated using the equations in *ASCE 7-16*.
C_p = external pressure coefficient. It is a fraction of the external pressure on the windward walls, leeward walls, side walls, and roof. The values of C_p are presented in Tables 2.6 and 2.7.

To compute the wind load that will be used for member design, combine the external and internal wind pressures, as follows:

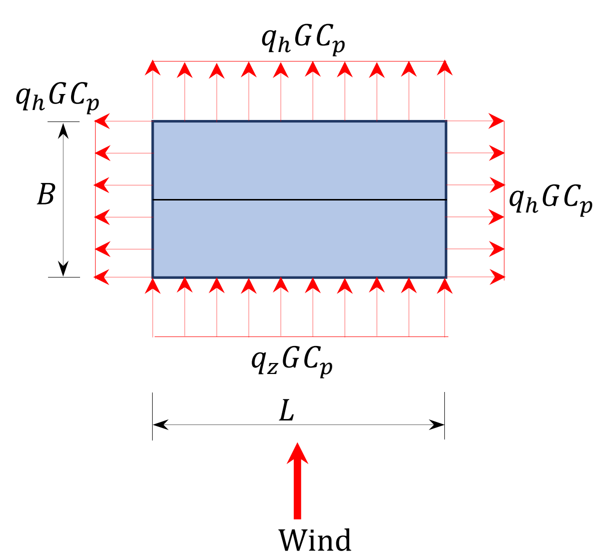
$$P = q_z G C_p - q_h \left(G C_{pi}\right) \tag{2.5}$$

where

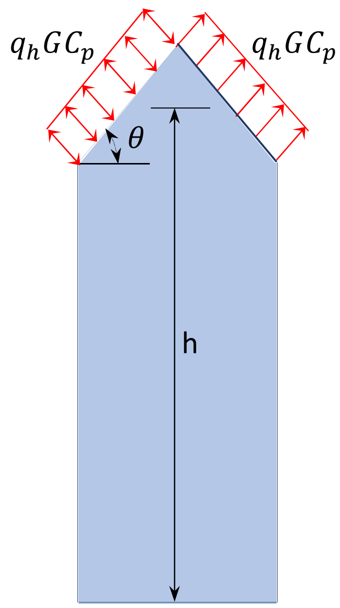
$G C_{pi}$ = the internal pressure coefficient from *ASCE 7-16*.



(a)



(b) Plan



(c) Elevation

Fig. 2.4. Typical wind distribution on a structure's walls and roof.

Table 2.6. Wall pressure coefficient, C_p, as specified in *ASCE 7-16*.

Surface	L/B	C_p	Use with
Windward wall	All values	0.8	q_z
Leeward wall	0 – 1 2 ≥ 4	-0.5 -0.3 -0.2	q_h
Side walls	All values	-0.7	q_h

Notes:

1. Positive and negative signs are indicative of the wind pressures acting toward and away from the surfaces.
2. L is the dimension of the building normal to the wind direction, and B is the dimension parallel to the wind direction.

Table 2.7. Roof pressure coefficients, C_p, for use with q_h, as specified in *ASCE 7-16*.

Wind direction	Windward angle, θ				Leeward angle, θ		
	h/L	10 °	15 °	20 °	10 °	15 °	≥20 °
Normal to ridge	≤ 0.25	-0.7	-0.5	-0.3	-0.3	-0.5	-0.6
	0.5	-0.9	-0.7	-0.4	-0.5	-0.5	-0.6
	> 1.0	-1.3	-1.0	-0.7	-0.7	-0.6	-0.6

Example 2.3

The two-story building shown in Figure 2.5 is an elementary school located on a flat terrain in a suburban area, with a wind speed of 102 mph and exposure category B. What is the wind velocity pressure at roof height for the main wind force resisting system (MWFRS)?

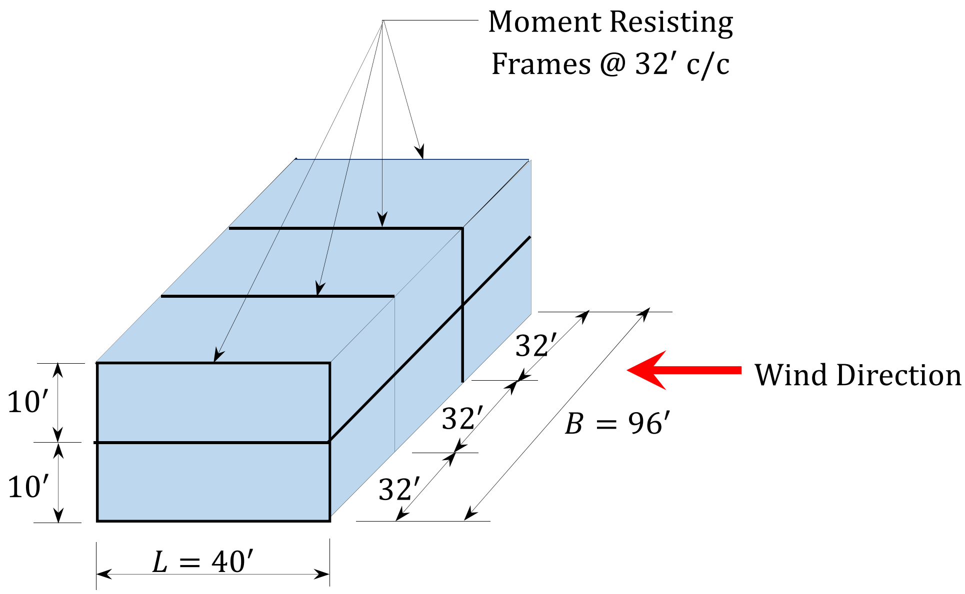


Fig. 2.5. Two – story building.

Solution

The mean height of the roof is $h = 20$ ft.
Table 26.10-1 from *ASCE 7-16* states that if the exposure category is B and the velocity pressure exposure coefficient for $h = 20'$, then $K_z = 0.7$.
The topography factor from section 26.8.2 of *ASCE 7-16* is $K_{zt} = 1.0$.
The wind directionality factor for MWFRS, according to Table 26.6-1 in *ASCE 7-16*, is $K_d = 0.85$.

Using equation 2.3, the velocity pressure at a roof height of $20'$ for the MWFRS is as follows:

$$q_z = 0.00256 K_z K_{zt} K_d V^2$$
$$= 0.00256(0.7)(1.0)(0.85)(102)^2 = 15.84 \text{ lb/ ft}^2$$

2.1.4.3 Snow Loads

In some geographic regions, the force exerted by accumulated snow and ice on buildings' roofs can be quite enormous, and it can lead to structural failure if not considered in structural design.

Suggested design values of snow loads are provided in codes and design specifications. The basis for the computation of snow loads is what is referred to as the ground snow load. The ground snow load is defined by the International Building Code (IBC) as the weight of snow on the ground surface. The ground snow loads for various parts of the United States can be obtained from the contour maps in *ASCE 7-16*. Some typical values of the ground snow loads from this standard are presented in Table 2.8. Once these loads for the required geographic areas have been established, they must be modified for specific conditions to obtain the snow load for structural design.

According to *ASCE 7-16*, the design snow loads for flat roofs and sloped roofs can be obtained using the following equations:

$$p_f = 0.7 C_e C_t I p_g$$
$$p_s = C_s p_f \tag{2.6}$$

where

p_f = design flat roof snow load.
p_s = design snow load for a sloped roof.
p_g = ground snow load.
I = importance factor. See Table 2.9 for importance factor values, depending on the category of the building.
C_e = exposure factor. See Table 2.10 for exposure factor values, depending on the terrain category.
C_t = thermal factor. See Table 2.11 for typical values.
C_s = slope factor. Values of C_s are provided in section 7.4.1 through 7.4.4 of *ASCE 7-16*, depending on various factors.

Table 2.8. Typical ground snow loads, as specified in *ASCE 7-16*.

Location	Load (PSF)
Lancaster, PA	30
Yakutat, AK	150
New York City, NY	30
San Francisco, CA	5
Chicago, IL	25
Tallahassee, FL	0

Table 2.9. Importance factor for snow load, I_s, as specified in *ASCE 7-16*.

Risk Category of Structure	Importance Factor
I	0.8
II	1.0
III	1.1
IV	1.2

Table 2.10. Exposure coefficient, C_e, as specified in *ASCE 7-16*.

Terrain Category	Exposure of Roof		
	Fully Exposed	Partially Exposed	Sheltered
A: Large city center	N/A	1.1	1.3
B: Urban and suburban areas	0.9	1.0	1.2
C: Open terrain with scattered obstructions	0.9	1.0	1.1
D: Unobstructed areas with wind over open water	0.8	0.9	1.0
Above the tree line in windswept mountainous areas	0.7	0.8	N/A
Alaska in areas with trees not within two miles of the site	0.7	0.8	N/A

Table 2.11. Thermal factor, C_t, as specified in *ASCE 7-16*.

Thermal Condition	Thermal Factor
All structures except as indicated below	1.0
Structures kept just above freezing and others with cold, ventilated roofs in which the thermal resistance (R-value) between the ventilated space and the heated space exceeds 25 $^\circ$F × h × ft^2/Btu (4.4 K × m^2/W)	1.1
Unheated and open air structures	1.2
Structures intentionally kept below freezing	1.3
Continuously heated greenhouses with a roof having a thermal resistance (R-value) less than 2.0 $^\circ$F × h × ft^2/Btu	0.85

Example 2.4

A single-story heated residential building located in the suburban area of Lancaster, PA is considered partially exposed. The roof of the building slopes at 1 on 20, and it is without overhanging eaves. What is the design snow load on the roof?

Solution

According to Figure 7.2-1 in *ASCE 7-16*, the ground snow load for Lancaster, PA is $p_g = 30$ psf.
Since 30 psf $>$ 20 psf, the rain-on-snow surcharge is not required.

To find the roof slope, use $\theta = \arctan\left(\frac{1}{20}\right) = 2.86°$.

According to *ASCE 7-16*, since $2.86° < 15°$, the roof is considered a low-slope roof. Table 7.3-2 in *ASCE 7-16* states that the thermal factor for a heated structure is $C_t = 1.0$ (see Table 2.11).

According to Table 7.3-1 in *ASCE 7-16*, the exposure factor for terrain category B, partially exposed is $C_e = 1.0$ (see Table 2.10).
Table 1.5-2 in *ASCE 7-16* states that the importance factor $I_s = 1.0$ for risk category II (see Table 2.9).

According to equation 2.6, the flat roof snow load is as follows:

$$p_f = 0.7C_eC_tIp_g$$

$$= (0.7)(1)(1)(1)(30 \text{ psf}) = 21 \text{ psf}$$

Since $21 \text{ psf} > 20I_s = (20 \text{ psf})(1) = 20 \text{ psf}$. Therefore, the design flat roof snow load is 21 psf.

2.1.4.4 Seismic Loads

The ground motion caused by seismic forces in many geographic regions of the world can be quite significant and often damages structures. This is particularly notable in regions near active geological faults. Thus, most building codes and standards require that structures be designed for seismic forces in such areas where earthquakes are likely to occur. The *ASCE 7-16* standard provides numerous analytical methods for estimating the seismic forces when designing structures. One of these methods of analysis, which will be described in this section, is referred to as the equivalent lateral force (ELF) procedure. The lateral base shear V and the lateral seismic force at any level computed by the ELF are shown in Figure 2.6. According to the procedure, the total static lateral base shear, V, in a specific direction for a building is given by the following expression:

$$V = \frac{S_{D1}}{T(R/I)}W \tag{2.7}$$

where

V = lateral base shear for the building. The estimated value of V must satisfy the following condition:

$$V_{min} = 0.044S_{DS}IW < V \leq V_{max} = \frac{S_{DS}W}{R/I} \tag{2.8}$$

W = effective seismic weight of the building. It includes total dead load of the building and its permanent equipment and partitions.

T = fundamental natural period of a building, which depends on the mass and the stiffness of the structure. It is computed using the following empirical formula:

$$T = C_t h_n^x \tag{2.9}$$

C_t = building period coefficient. The value of $C_t = 0.028$ for structural steel moment resisting frames, 0.016 for reinforced concrete rigid frames, and 0.02 for most other structures (see Table 2.12).

h_n = height of the highest level of the building, and $x = 0.8$ for steel rigid moment frames, 0.9 for reinforced concrete rigid frames, and 0.75 for other systems.

Table 2.12. C_t values for various structural systems.

Structural System	C_t	x
Steel moment resisting frames	0.028	0.8
Eccentrically braced frames (EBF)	0.03	0.75
All other structural systems	0.02	0.75

S_{DI} = design spectral acceleration. It is estimated by using a seismic map that provides an earthquake's intensity of design for structures at locations with $T = 1$ second.

S_{Ds} = design spectral acceleration. It is estimated by using a seismic map that provides an earthquake's intensity of design for structures with $T = 0.2$ second.

R = response modification coefficient. It accounts for the ability of a structural system to resist seismic forces. The values of R for several common systems are presented in Table 2.13.

I = importance factor. This is a measure of the consequences to human life and damage to property in the event that the structure fails. The value of the importance factor is 1 for office buildings, but equals 1.5 for hospitals, police stations, and other public buildings where loss of more life or damages to property are anticipated should a structure fail.

Table 2.13. Response modification coefficient, R, as specified in *ASCE 7-16*.

Seismic Force-Resisting System	R
Bearing wall systems	
Ordinary reinforced concrete shear walls	4
Ordinary reinforced masonry shear walls	2
Light-frame (cold-formed steel) walls sheathed with structural panels rated for shear resistance or steel sheets	$6\frac{1}{2}$
Building frame systems	
Ordinary reinforced concrete shear walls	5
Ordinary reinforced masonry shear walls	2
Steel buckling-restrained braced frames	8
Moment-resisting frame systems	
Steel special moment frames	8
Steel ordinary moment frames	$3\frac{1}{2}$
Ordinary reinforced concrete moment frames	3

Once the total seismic static lateral base shear force in a given direction for a structure has been computed, the next step is to determine the lateral seismic force that will be applied to each floor level using the following equation:

$$F_x = \frac{W_x h_x^k}{\sum W_i h_i^k} V \tag{2.10}$$

where

F_x = lateral seismic force applied to level x.
w_i and w_x = effective seismic weights at levels i and x.
h_i and h_x = heights from the base of the structure to floors at levels i and x.
$\sum W_i h_i^k$ = summation of the product W_i and h_i^k over the entire structure.
k = distribution exponent related to the fundamental natural period of the structure. For $T \leq$ 0.5s, k = 1.0, and for $T \geq$ 2.5s, k = 2.0. For T lying between 0.5s and 2.5s, k can be computed using the following relationship:

$$k = 1 + \frac{T - 0.5}{2} \tag{2.11}$$

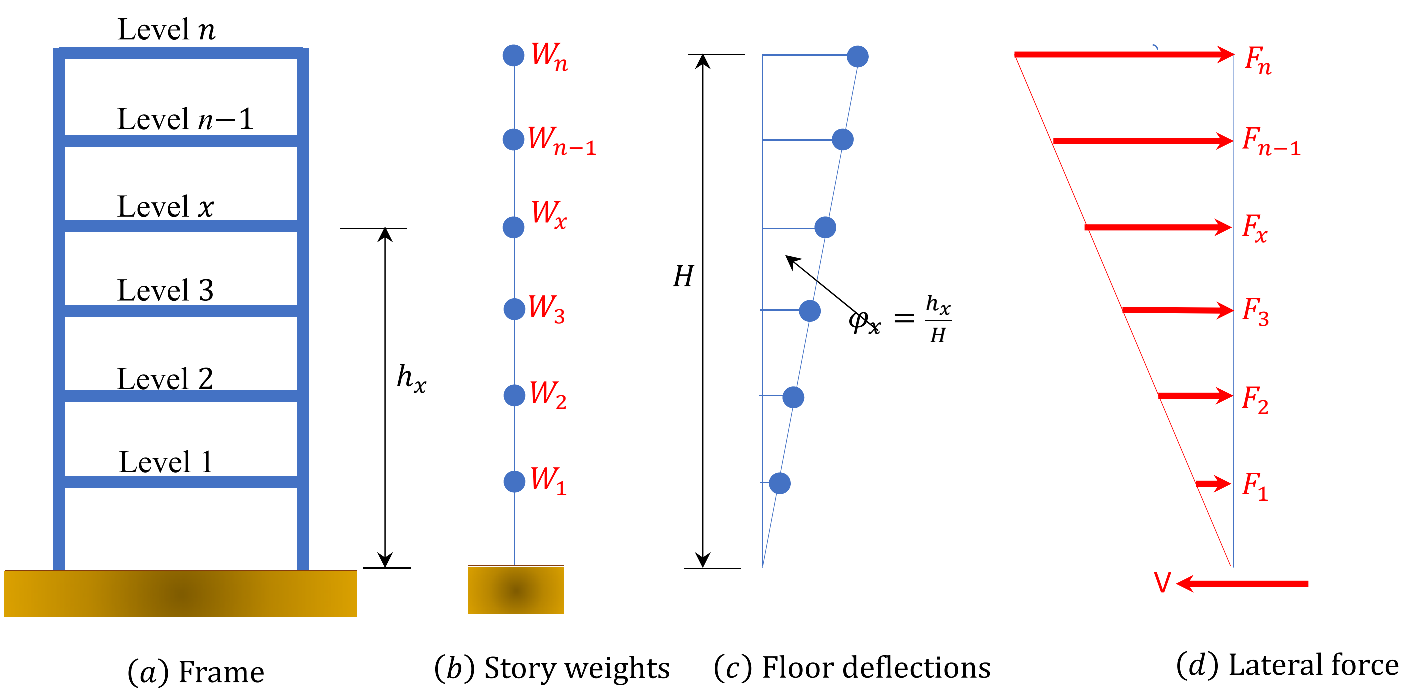


Fig. 2.6. Equivalent lateral force procedure

Example 2.5

The five-story office steel building shown in Figure 2.7 is laterally braced with steel special moment resisting frames, and it measures 75 ft by 100 ft in the plan. The building is located in New York City. Using the *ASCE 7-16* equivalent lateral force procedure, determine the lateral force that will be applied to the fourth floor of the structure. The roof dead load is 32 psf, the floor dead load (including the partition load) is 80 psf, and the flat roof snow load is 40 psf. Ignore the weight of cladding. The design spectral acceleration parameters are $S_{DS} = 0.28$, and $S_{D1} = 0.11$.

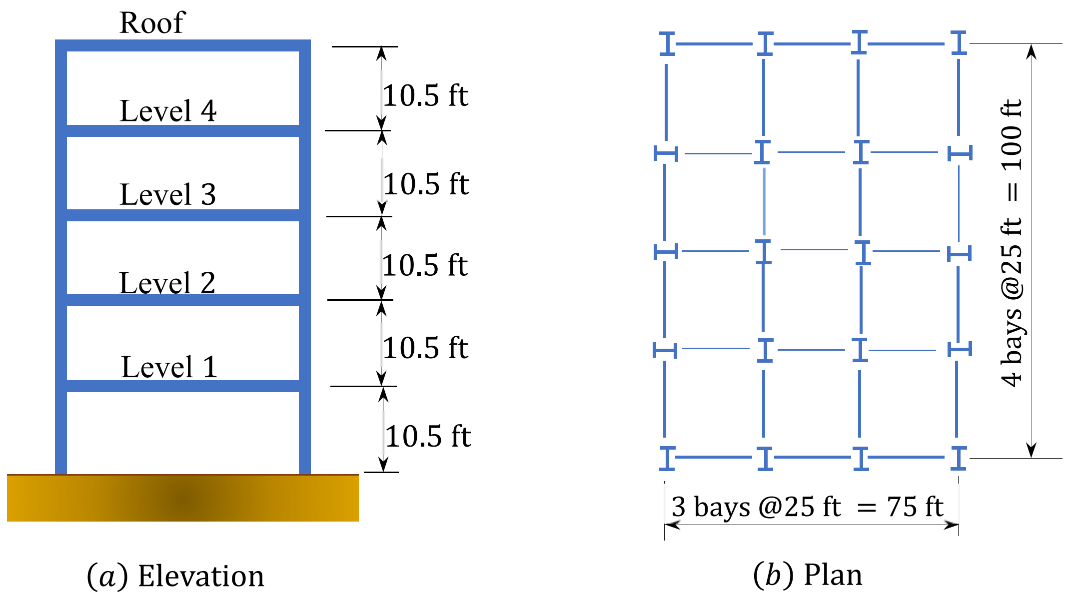


Fig. 2.7. Five – story office building.

Solution

$S_{DS} = 0.28$ and $S_{D1} = 0.11$ (given).

$R = 8$ for special moment resisting steel frame (see Table 2.13).

An office building is in occupancy risk category II, so $I_e = 1.0$ (see Table 2.9).

Calculate the approximate fundamental natural period of the building T_a.

$C_t = 0.028$ and $x = 0.8$ (from Table 2.12 for steel moment resisting frames).

$h_n = \text{Roof height} = 52.5 \text{ ft}$

$T = T_a = C_t(h_n^x) = 0.028(52.5^{0.8}) = 0.67 \text{ sec} < T_L = 6s$ (Figure 22-14 in *ASCE 7-16*).

Determine the dead load at each level. Since the flat roof snow load given for the office building is greater than 30 psf, 20% of the snow load must be included in the seismic dead load computations.

The weight assigned to the roof level is as follows:

$$W_{\text{roof}} = (32 \text{ psf})(75 \text{ ft})(100 \text{ ft}) + (20\%)(40\text{psf})(75 \text{ ft})(100 \text{ ft}) = 300{,}000 \text{ lb}$$

The weight assigned to all other levels is as follows:

$$W_i = (80 \text{ psf})(75\text{ft})(100 \text{ ft}) = 600{,}000 \text{ lb}$$

The total dead load is as follows:

$$W_{Total} = 300{,}000 \text{ lb} + (4)(600{,}000 \text{ lb}) = 2700 \text{ k}$$

Calculate the seismic response coefficient C_s.

$$C_s = \frac{S_{DS}}{R/I_e} = \frac{0.28}{8/1.0} = 0.035$$

$$\leq \frac{S_{D1}}{\left(\frac{TR}{I_e}\right)} = \frac{0.11}{[(0.67)(8)/1.0]} = 0.021$$

Therefore, $C_s = 0.021 > 0.01$

Determine the seismic base shear V.

$$V = C_s W = (0.021)(2700 \text{ kips}) = 56.7\text{k}$$

Calculate the lateral force applied to the fourth floor.

$$k = 1 + \frac{T-0.5}{2} = 1 + \frac{0.67-0.5}{2} = 1.085$$

$$F_4 = \frac{w_4 h_4^k}{\sum_{i=1}^{m} w_i h_i^k}(V)$$

$$= \frac{600(42)^{1.085}}{600(10.5)^{1.085}+600(21)^{1.085}+600(31.5)^{1.085}+600(42)^{1.085}+300(52.5)^{1.085}}(56.7 \text{ k})$$

$$= 18.51 \text{ k}$$

2.1.4.5 Hydrostatic and Earth Pressures

Retaining structures must be designed against overturning and sliding caused by hydrostatic and earth pressures to ensure the stability of their bases and walls. Examples of retaining walls include gravity walls, cantilever walls, counterfort walls, tanks, bulkheads, sheet piles, and others.
The pressures developed by the retained material are always normal to the surfaces of the retaining structure in contact with them, and they vary linearly with height. The intensity of normal pressure, p, and the resultant force, P, on the retaining structure is computed as follows:

$$p = \gamma h$$

$$P = \frac{1}{2}\gamma h^2 \quad (2.12)$$

Where

γ = unit weight of the retained material.
h = distance from the surface of the retained material and the point under consideration.

2.1.4.6 Miscellaneous Loads

There are numerous other loads that may also be considered when designing structures, depending on specific cases. Their inclusion in the load combinations will be based on a designer's discretion if they are perceived to have a future significant impact on structural integrity. These loads include thermal forces, centrifugal forces, forces due to differential settlements, ice loads, flooding loads, blasting loads, and more.

2.2 Load Combinations for Structural Design

Structures are designed to satisfy both strength and serviceability requirements. The strength requirement ensures the safety of life and property, while the serviceability requirement guarantees the comfortability of occupancy (people) and the aesthetics of the structure. To meet the afore-stated requirements, structures are designed for the critical or the largest load that would act on them. The critical load for a given structure is found by combining all the various possible loads that a structure may carry during its lifetime. Sections 2.3.1 and 2.4.1 of *ASCE 7-16* provide the following load combinations for use when designing structures by the Load and Resistance Factor Design (LRFD) and the Allowable Strength Design (ASD) methods.

For LRFD, the load combinations are as follows:

1. $1.4D$
2. $1.2\,D + 1.6L + 0.5(L_r or S\ or\ R)$
3. $1.2D + 1.6(L_r\ or\ S\ or\ R) + (L\ or\ 0.5W)$
4. $1.2D + 1.0W + L + 0.5(L_r\ or\ S\ or\ R)$
5. $0.9D + 1.0W$

For ASD, the load combinations are as follows:

1. D
2. $D + L$
3. $D + (L_r\ or\ S\ or\ R)$
4. $D + 0.75L + 0.75(L_r\ or\ S\ or\ R)$
5. $D + (0.6W)$

where

D = dead load.

L = live load due to occupancy.
L_r = roof live load.
S = snow load.
R = nominal load due to initial rainwater or ice, exclusive of the ponding contributions.
W = wind load.
E = earthquake load.

Example 2.6

A floor system consisting of wooden joists spaced 6 ft apart on the center and a tongue and groove wood boarding, as shown in Figure 2.8, supports a dead load (including the weight of the beam and boarding) of 20 psf and a live load of 30 psf. Determine the maximum factored load in lb/ft that each floor joist must support using the LRFD load combinations.

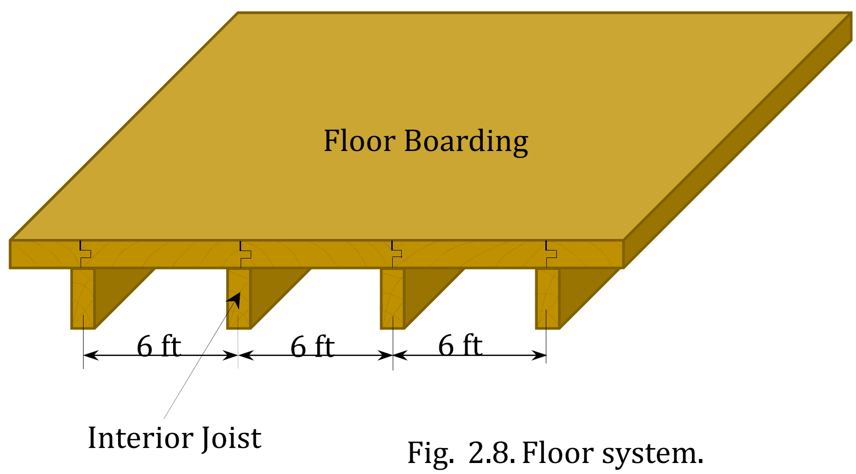


Fig. 2.8. Floor system.

Solution

Dead load $D = (6)(20) = 120$ lb/ft
Live load $L = (6)(30) = 180$ lb/ft

Determining the maximum factored loads W_u using the LRFD load combinations and neglecting the terms that have no values, yields the following:

$$W_u = (1.4)(120) = 168 \text{ lb/ft}$$

$$W_u = (1.2)(120) + (1.6)(180) = 288 \text{ lb/ft}$$

$$W_u = (1.2)(120) + (0.5)(180) = 234 \text{ lb/ft}$$

$$W_u = (1.2)(120) + (0.5)(180) = 234 \text{ lb/ft}$$

$$W_u = (1.2)(120) + (0.5)(180) = 234 \text{ lb/ft}$$

$$W_u = (0.9)(120) = 108 \text{ lb/ft}$$

The governing factored load = 288 lb/ft

2.3 Tributary Width and Area

A tributary area is the area of loading that will be sustained by a structural member. For example, consider the exterior beam B1 and the interior beam B2 of the one-way slab system shown in Figure 2.9. The tributary width for B1 is the distance from the centerline of the beam to half the distance to the next or adjacent beam, and the tributary area for the beam is the area bordered by the tributary width and the length of the beam, as shaded in the figure. For the interior beam B2-B3, the tributary width W_T is half the distance to the adjacent beams on both sides.

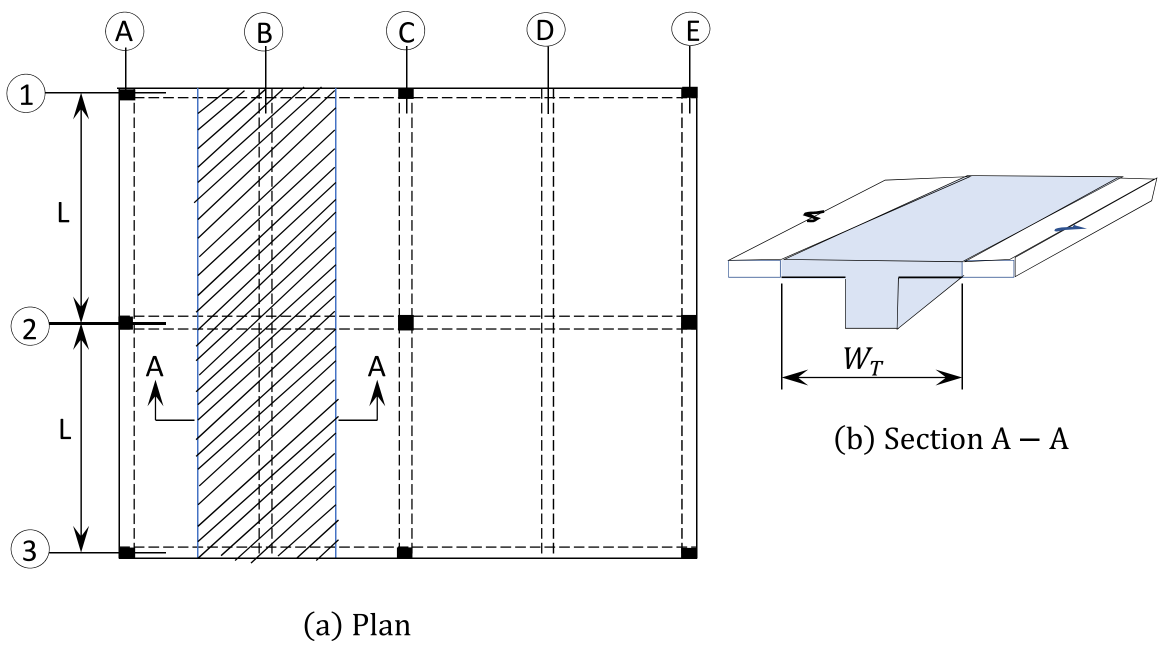


Fig. 2.9. Tributary area.

2.4 Influence Areas

Influence areas are areas of loading that influence the magnitude of loads carried by a particular structural member. Unlike tributary areas, where the load within an area is sustained by the member, all the loads in the influence area are not supported by the member under consideration.

2.5 Live Load Reduction

Most codes and standards allow for reduction in live loads when designing large floor systems, since it is very unlikely that such systems will always support the estimated maximum live loads at every instance. Section 4.7.3 of *ASCE 7-16* permits a reduction of live loads for members that have an influence area of $A_I \geq 37.2\ \text{m}^2 (400\ \text{ft}^2)$. The influence area is the product of the tributary area and the live load element factor. The *ASCE 7-16* equations for determining the reduced live load based on the influence area are as follows:

$$L = L_0\left(0.25 + \frac{15}{\sqrt{K_{LL}A_T}}\right)\ \text{(FPS units)}$$

$$L = L_0\left(0.25 + \frac{4.57}{\sqrt{K_{LL}A_T}}\right)\ \text{(SI units)} \tag{2.13}$$

where

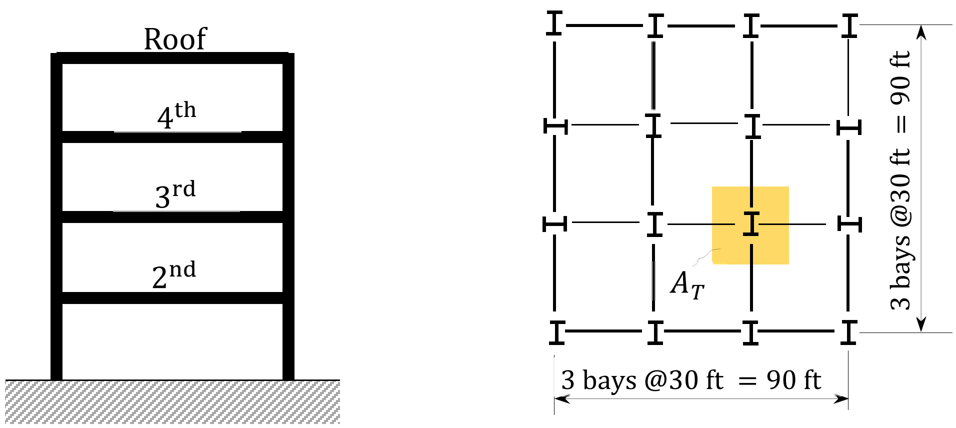
L = reduced design live load per ft^2 (or m^2).
$\geq 0.50\ L_o$ for structural members supporting one floor (e.g. beams, girders, slabs, etc.).
$\geq 0.40\ L_o$ for structural members supporting two or more floors (e.g. columns, etc.).
No reduction is permitted for floor live loads greater than $4.79\ \text{kN/m}^2 (100\ \text{lb/ft}^2)$ or for floors of public assembly, such as stadiums, auditoriums, movie theaters, etc., as there is a greater possibility of such floors being overloaded or used as car garages.
L_o = unreduced design live load per ft^2 (or m^2) from Table 2.2 (Table 4.3-1 in *ASCE 7-16*).
A_T = tributary area of member in ft^2 (or m^2).
$K_{LL} = A_I/A_T$ = live load element factor from Table 2.14 (see values tabulated in Table 4.7-1 in *ASCE 7-16*).
$A_I = K_{LL}A_T$ = influence area.

Table 2.14. Live load element factor.

Building Element	K_{LL}
Interior columns and exterior columns without cantilever slabs	4
Exterior columns with cantilever slabs	3
Corner columns with cantilever slabs	2
Interior beams and edge beams without cantilever slabs	2
All other members, including panels in two-way slabs	1

Example 2.7

A four-story school building used for classrooms has its columns spaced as shown in Figure 2.10. The flat roof loading of the structure is estimated to be $25\ \text{lb/ft}^2$. Determine the reduced live load supported by an interior column at the ground level.



(b) Plan

(a) Elevation

Fig. 2.10. A four − story schol building.

Solution

Any interior column at the ground level supports the roof load and the live loads on the second, third, and fourth floors.

The tributary area of an interior column is $A_T = (30 \text{ ft})(30 \text{ ft}) = 900 \text{ ft}^2$

The roof live load is $F_R = (25 \text{lb/ft}^2)(900 \text{ ft}^2) = 22{,}500 \text{ lb} = 22.5 \text{ k}$

For the floor live loads, use the *ASCE 7-16* equations to check for the possibility of a reduction.

$L_o = 40 \text{ lb/ft}^2$ (from Table 4.1 in *ASCE 7-16*).

If the interior column $K_{LL} = 4$, then the influence area $A_1 = K_{LL} A_T = (4)(900 \text{ ft}^2) = 3600 \text{ ft}^2$.

Since $3600\text{ft}^2 > 400\text{ft}^2$, the live load can be reduced using equation 2.14, as follows:

$$L = L_0 \left(0.25 + \frac{15}{\sqrt{K_{LL} A_T}}\right) = 40 \left(0.25 + \frac{15}{\sqrt{3600}}\right) = 20 \text{ lb/ft}^2$$

According to Table 4.1 in *ASCE 7-16*, the reduced load as a fraction of the unreduced floor live load for a classroom is $\left(\frac{20}{40}\right) = 0.50 > 0.4$. Thus, the reduced floor live load is as follows:

$$F_F = (20 \text{ lb/ft}^2)(900\text{ft}^2) = 18{,}000\text{lb} = 18 \text{ k}$$

The total load supported by the interior column at the ground level is as follows:

$$F_{Total} = 22.5 \text{ k} + 3(18 \text{ k}) = 76.5 \text{ k}$$

Chapter Summary

Structural loads and loading systems: Structural elements are designed for the worst possible load combinations. Some of the loads that could act on a structure are briefly defined below.

Dead loads: These are loads of a constant magnitude in a structure. They include the weight of structure and the loads that are permanently attached to the structure.

Live loads: These are loads of varying magnitudes and positions. They include moveable loads and loads due to occupancy.

Impact loads: Impact loads are sudden or rapid loads applied on a structure over a relatively short period of time compared with other structural loads.

Rain loads: These are loads due to accumulation of water on a roof top after a rainstorm.

Wind loads: These are loads due to wind pressure exerted on structures.

Snow loads: These are loads exerted on a structure by accumulated snow on a rooftop.

Earthquake loads: These are loads exerted on a structure by the ground motion caused by seismic forces.

Hydrostatic and earth pressures: These are loads on retaining structures due to pressures developed by the retained materials. They vary linearly with the height of the walls.

Load combinations: The two building design methods are the Load and Resistance Factor Design method (LRFD) and the Allowable Strength Design method (ASD). Some of the load combinations for these methods are shown below.

LRFD:

1. $1.4D$
2. $1.2\,D + 1.6L + 0.5(L_r or S\ or\ R)$
3. $1.2D + 1.6(L_r\ or\ S\ or\ R) + (L\ or\ 0.5W)$
4. $1.2D + 1.0W + L + 0.5(L_r\ or\ S\ or\ R)$
5. $0.9D + 1.0W$

ASD:

1. D
2. $D + L$
3. $D + (L_r \; or \; S \; or \; R)$
4. $D + 0.75L + 0.75(L_r \; or \; S \; or \; R)$
5. $D + (0.6W)$

References

ACI (2016), Building Code Requirements for Structural Concrete (ACI 318-14), American Concrete Institute.

ASCE (2016), Minimum Design Loads for Buildings and Other Structures, ASCE 7-16, ASCE.

ICC (2012), International Building Code, International Code Council.

Practice Problems

2.1 Determine the maximum factored moment for a roof beam subjected to the following service load moments:

$M_D = 40 \; psf$ (dead load moment)

$M_{L_r} = 36 \; psf$ (roof live load moment)

$M_s = 16 \; psf$ (snow load moment)

2.2 Determine the maximum factored load sustained by a column subjected to the following service loads:

$P_D = 500 \; kips$ (dead load)

$P_L = 280 \; kips$ (floor live load)

$P_S = 200 \; kips$ (snow load)

$P_E = \pm 30\ kips$ (earthquake load)

$P_w = \pm 70\ kips$ (wind load)

2.3 The typical layout of a steel-reinforced concrete composite floor system of a library building is shown in Figure P2.1. Determine the dead load in lb/ft acting on a typical interior beam $B1$-$B2$ in the second floor. All beams are $W12 \times 44$, spaced at 10 ft o.c. The distributed loads on the second floor are as follows:

2 in. thick sand-cement screed	= 0.25 psf
6 in. thick reinforced concrete slab	= 50 psf
Suspended metal lath and gypsum plaster ceiling	= 10 psf
Electrical and mechanical services	= 4 psf

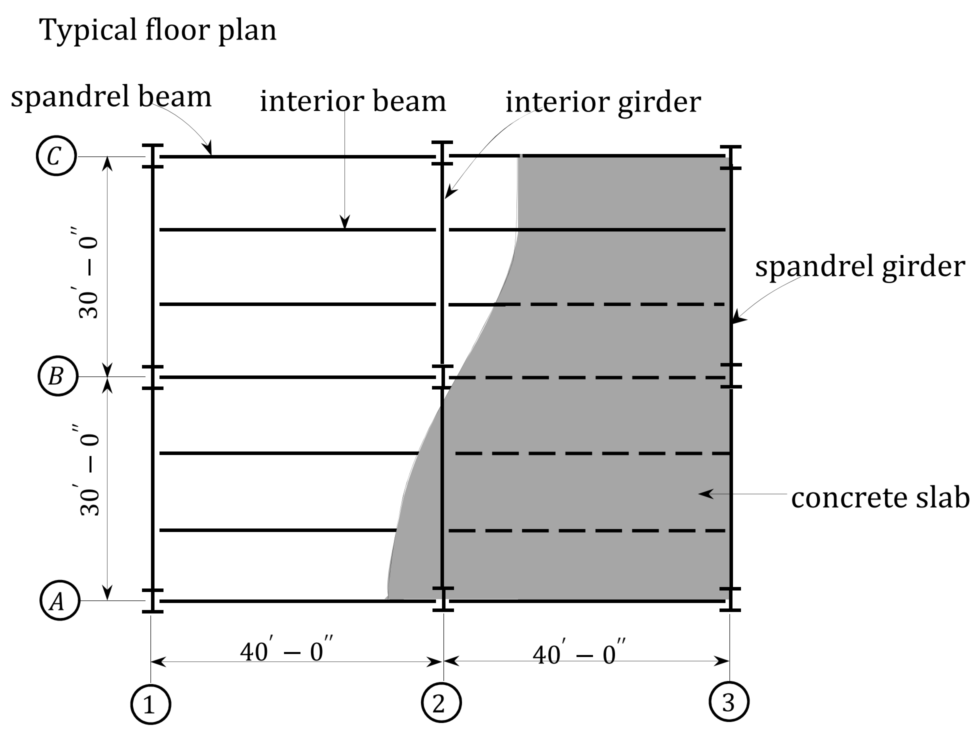


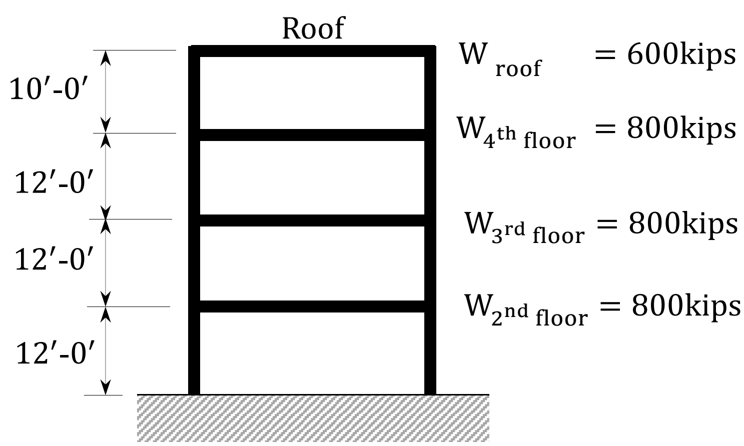
Fig. P2.1. A steel − reinforced concrete composite floor system.

2.4 The second-floor layout of an elementary school building is shown in Figure P2.1. The floor finishing is similar to that of practice problem 2.3, with the exception that the ceiling is an acoustical fiberboard of a minimum design load of 1 psf. All beams are $W12 \times 75$, with a weight of 75 lb/ft, and all girders are $W16 \times 44$, with a self-weight of 44 lb/ft. Determine the dead load on a typical interior girder $A2$-$B2$.

2.5 The second-floor layout of an office facility is shown in Figure P2.1. The floor finishing is similar to that of practice problem 2.3. Determine the total dead load applied to the interior column $B2$ at the second floor. All beams are $W14 \times 75$, and all girders are $W18 \times 44$.

2.6 A four-story flat roof hospital building shown in Figure P2.2 has concentrically braced frames as its lateral force resisting system. The weight at each floor level is indicated in the figure. Determine the seismic base shear in kips given the following design data:

$S_1 = 1.5\text{g}$
$S_s = 0.6\text{g}$
Site class = D



(a) Elevation

Fig. P2.2. A four – story flat roof building.

2.7 Use *ASCE 7-16* to determine the snow load (psf) for the building shown in Figure P2.3.
The following data apply to the building:

Ground snow load = 30 psf
Roof is fully exposed with asphalt shingles.
Roof's slope angle = 25°
Open terrain
Occupancy Category I
Unheated structure

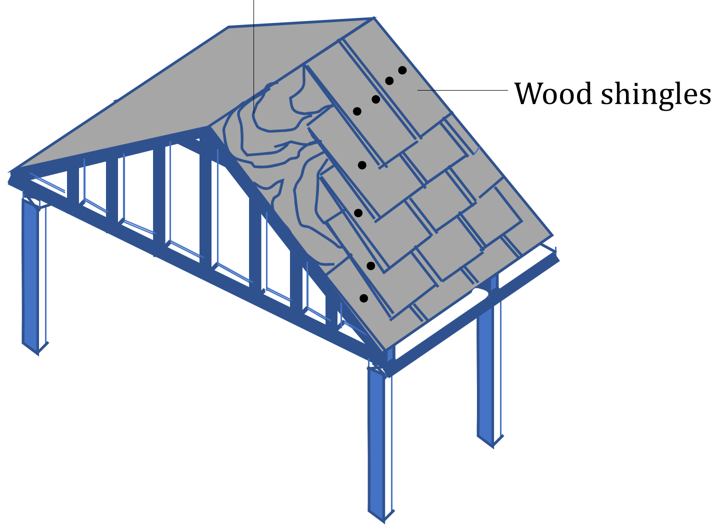


Fig. P2.3. A sample roof.

2.8. In addition to the design snow load computed in practice problem 2.7, the roof of the building in Figure P2.3 is subjected to a dead load of 16 psf (including the weight of a truss, roof board, and asphalt shingle) on the horizontal plane. Determine the uniform load acting on the interior truss, if the trusses are 6ft-0in on center.

2.9 Wind blows at a speed of 90 mph on the enclosed storage facility shown in Figure P2.4. The facility is situated on a flat terrain with an exposure category B. Determine the wind velocity pressure in psf at the eave height of the facility. The topographic factor is $K_{zt} = 1.0$.

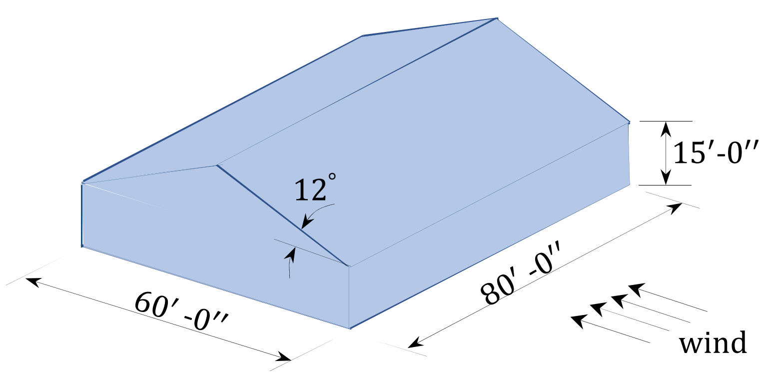


Fig. P2.4. An enclosed storgae facility.

PART TWO
ANALYSIS OF STATICALLY DETERMINATE STRUCTURES

Chapter 3

Equilibrium Structures, Support Reactions, Determinacy and Stability of Beams and Frames

3.1 Equilibrium of Structures

Engineering structures must remain in equilibrium both externally and internally when subjected to a system of forces. The equilibrium requirements for structures in two and three dimensions are stated below.

3.1.1 Equilibrium in Two Dimensions

For a structure subjected to a system of forces and couples which are lying in the xy plane to remain at rest, it must satisfy the following three equilibrium conditions:

$$\sum F_x = 0; \sum F_y = 0; \sum M_z = 0 \tag{3.1}$$

The above three conditions are commonly referred to as the equations of equilibrium for planar structures. $\sum F_x$ and $\sum F_y$ are the summation of the x and y components of all the forces acting on the structure, and $\sum M_z$ is the summation of the couple moments and the moments of all the forces about an axis z, perpendicular to the plane xy of the action of the forces.

3.1.2 Equilibrium in Three Dimensions

A structure in three dimensions, that is, in a space, must satisfy the following six requirements to remain in equilibrium when acted upon by external forces:

$$\begin{aligned} &\sum F_x = 0; \sum F_y = 0; \sum F_z = 0 \\ &\sum M_x = 0; \sum M_y = 0; \sum M_z = 0 \end{aligned} \tag{3.2}$$

3.2 Types of Supports and Their Characteristics

The type of support provided for a structure is important in ensuring its stability. Supports connect the member to the ground or to some other parts of the structure. It is assumed that the student is already familiar with several types of supports for rigid bodies, as this was introduced in the statics course. However, the characteristics of some of the supports are described below and shown in Table 3.1.

3.2.1 Pin or Hinge Support

A pin support allows rotation about any axis but prevents movement in the horizontal and vertical directions. Its idealized representation and reactions are shown in Table 3.1.

3.2.2 Roller Support

A roller support allows rotation about any axis and translation (horizontal movement) in any direction parallel to the surface on which it rests. It restrains the structure from movement in a vertical direction. The idealized representation of a roller and its reaction are also shown in Table 3.1.

3.2.3 Rocker Support

The characteristics of a rocker support are like those of the roller support. Its idealized form is depicted in Table 3.1.

3.2.4 Link

A link has two hinges, one at each end. It permits movement in all direction, except in a direction parallel to its longitudinal axis, which passes through the two hinges. In other words, the reaction force of a link is in the direction of the link, along its longitudinal axis.

3.2.5 Fixed Support

A fixed support offers a constraint against rotation in any direction, and it prevents movement in both horizontal and vertical directions.

3.3 Determinacy and Stability of Beams and Frames

Prior to the choice of an analytical method, it is important to establish the determinacy and stability of a structure. A determinate structure is one whose unknown external reaction or internal members can be determined using only the conditions of equilibrium. An indeterminate structure is one whose unknown forces cannot be determined by the conditions of static equilibrium alone and will require, in addition, a consideration of the compatibility conditions of different parts of the structure for its complete analysis. Furthermore, structures must be stable to be able to serve their desirable functions. A structure is considered stable if it maintains its geometrical shape when subjected to external forces.

3.3.1 Formulations for Stability and Determinacy of Beams and Frames

The conditions of determinacy, indeterminacy, and instability of beams and frames can be stated as follows:

$$3m + r < 3j + C \quad \text{Structure is statically unstable}$$

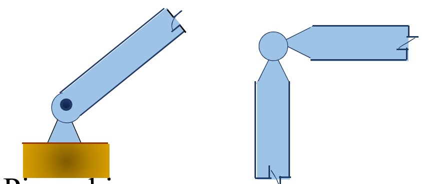
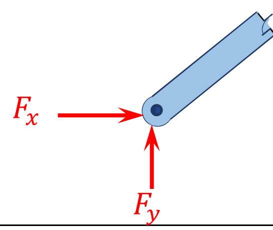
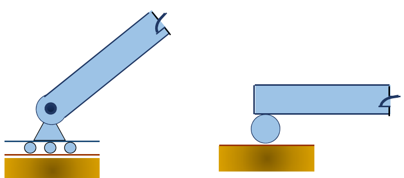
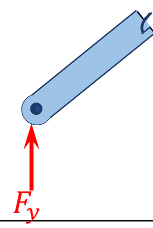
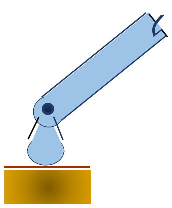
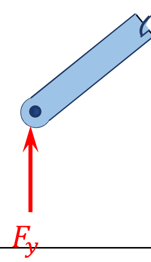
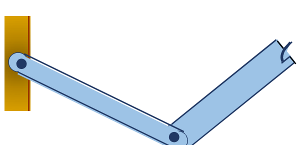
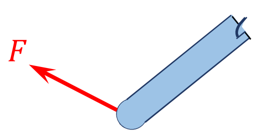
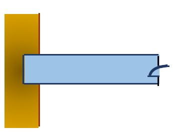
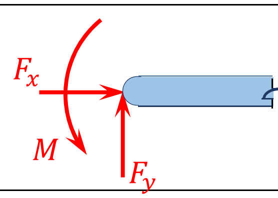
$$3m + r = 3j + C \quad \text{Structure is statically determinate}$$

$$3m + r > 3j + C \quad \text{Structure is statically indeterminate} \tag{3.3}$$

where

r = number of support reactions.
C = equations of condition (two equations for one internal roller and one equation for each internal pin).
m = number of members.
j = number of joints.

Table 3.1. Types of supports.

Idealization of Support	Reaction	Characteristics
 Pin or hinge		Prevents movement in the vertical and horizontal direction but allows rotation.
 Roller		Prevents movement in the vertical direction but allows rotation and translation in the horizontal direction.
 Rocker		The characteristics of a rocker support are similar to that of a roller.
 Link		Prevents movement in the direction perpendicular to the axis of the link.
 Fixed		Does not allow translation in any direction and rotation.

3.3.2 Alternative Formulation for Determinacy and Stability of Beams and Frames

$$r + F_i < 3m \quad \text{Structure is statically unstable}$$

$$r + F_i = 3m \quad \text{Structure is statically determinate} \tag{3.4}$$

$$r + F_i > 3m \quad \text{Structure is statically indeterminate}$$

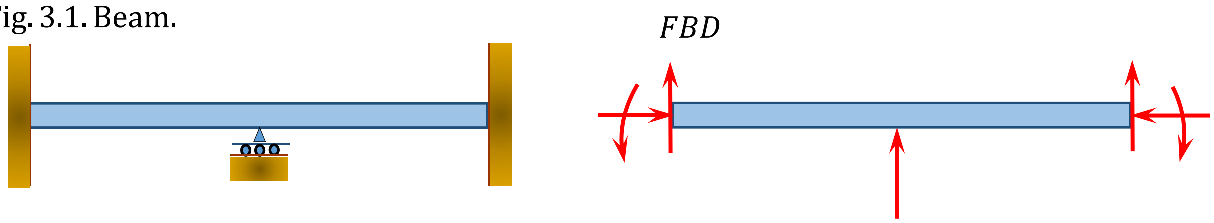
where

r = number of support reactions.
F_i = number of reaction forces transmitted by an internal hinge or internal roller.
m = number of members.

Example 3.1

Classify the beams shown in Figure 3.1 through Figure 3.5 as stable, determinate, or indeterminate, and state the degree of indeterminacy where necessary.

Fig. 3.1. Beam.



Solution

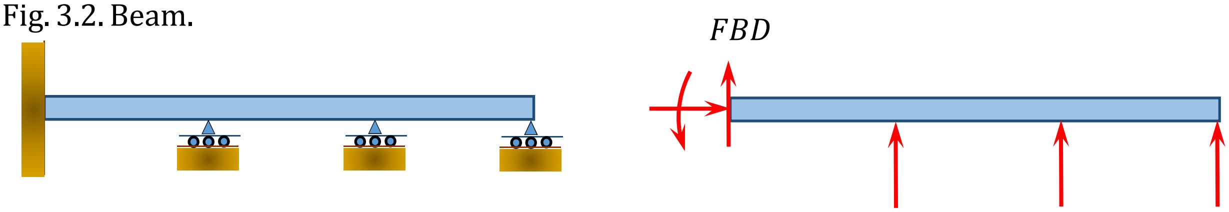
First, draw the free-body diagram of each beam. To determine the classification, apply equation 3.3 or equation 3.4.

Using equation 3.3, $r = 7,\ m = 2,\ c = 0,\ j = 3$. Applying the equation leads to $3(2) + 7 > 3(3) + 0$, or $13 > 9$. Therefore, the beam is statically indeterminate to the 4°.

Using equation 3.4, $r = 7,\ m = 1,\ F_i = 0$. Applying the equation leads to $7 + 0 > (3)(1)$, or $7 > 3$. Therefore, the beam is statically indeterminate to the 4°.

Note: When using equation 3.3, the portions on either side of the interior support are counted as separate members.

Fig. 3.2. Beam.

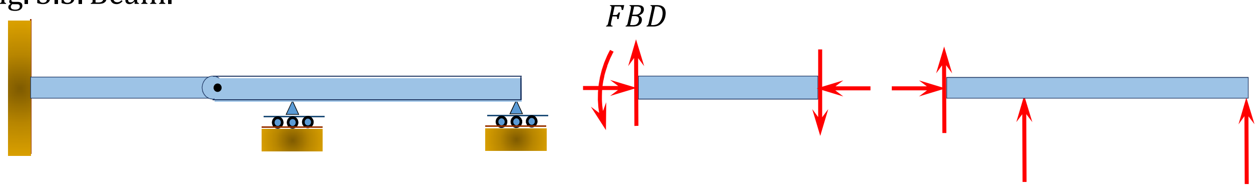


Solution

Using equation 3.3, $r = 6$, $m = 3$, $c = 0$, $j = 4$. Applying the equation leads to $3(3) + 6 > 3(4) + 0$, or $15 > 12$. Therefore, the beam is statically indeterminate to the 3°.

Using equation 3.4, $r = 6$, $m = 1$, $F_i = 0$. Applying the equation leads to $6 + 0 > (3)(1)$, or $6 > 3$. Therefore, the beam is statically indeterminate to the 3°.

Fig. 3.3. Beam.

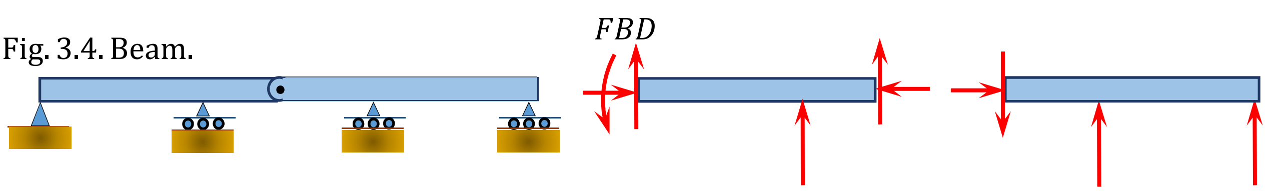


Solution

Using equation 3.3, $r = 5$, $m = 3$, $c = 1$, $j = 4$. Applying the equation leads to $3(3) + 5 > 3(4) + 1$, or $14 > 13$. Therefore, the beam is statically indeterminate to the 1°.

Using equation 3.4, $r = 5$, $m = 2$, $F_i = 2$. Applying the equation leads to $5 + 2 > 3(2)$, or $7 > 6$. Therefore, the beam is statically indeterminate to the 1°.

Fig. 3.4. Beam.

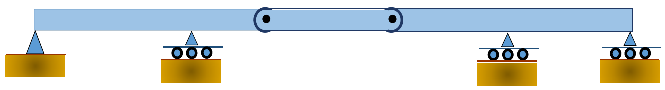


Solution

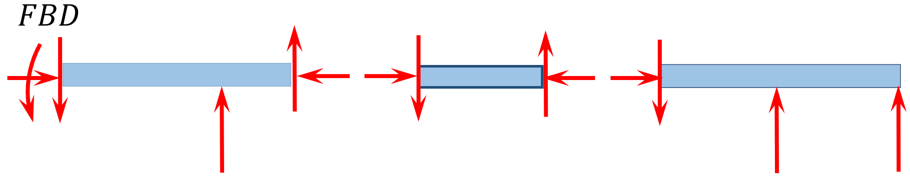
Using equation 3.3, $r = 5$, $m = 4$, $c = 1$, $j = 5$. Applying the equation leads to $3(4) + 5 > 3(5) + 1$, or $17 > 16$. Therefore, the equation is statically indeterminate to the 1°.

Using equation 3.4, $r = 5,\ m = 2,\ F_i = 2$. Applying the equation leads to $5 + 2 > 3(2)$, or $7 > 6$. Therefore, the beam is statically indeterminate to the 1°.

Fig. 3.5. Beam.



FBD



Solution

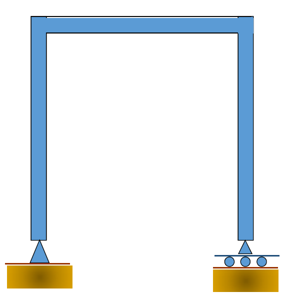
Using equation 3.3, $r = 5,\ m = 5,\ c = 2,\ j = 6$. Applying the equation leads to $3(5) + 5 = 3(6) + 2$, or $20 = 20$. Therefore, the beam is statically determinate.

Using equation 3.4, $r = 5,\ m = 3, F_i = 4$. Applying the equation leads to $5 + 4 > 3(3)$, or $9 = 9$. Therefore, the beam is statically determinate.

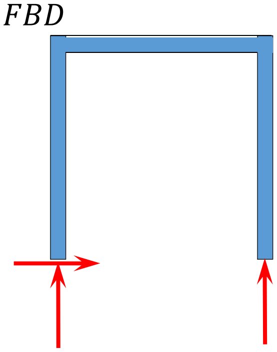
Example 3.2

Classify the frames shown in Figure 3.6 through Figure 3.8 as stable or unstable and determinate or indeterminate. If indeterminate, state the degree of indeterminacy.

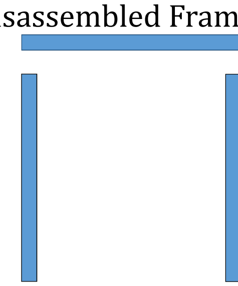
Fig. 3.6. Frame.



FBD



Disassembled Frame



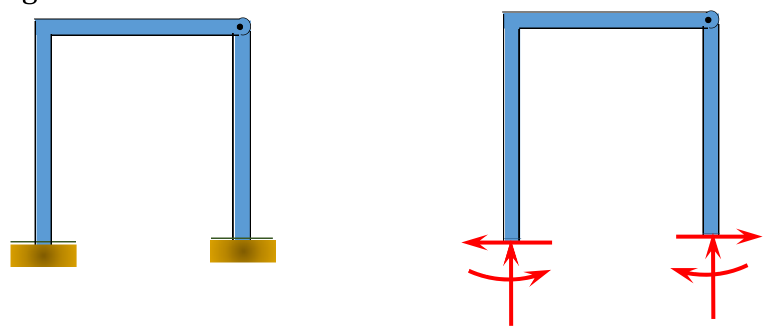
Solution

Using equation 3.3, $r = 3,\ m = 3,\ c = 0,\ j = 4$. Applying the equation leads to $3(3) + 3 = 3(4) + 0$, or $12 = 12$. Therefore, the frame is statically determinate.

Using equation 3.4, $r = 3$, $m = 1$, $F_i = 0$. Applying the equation leads to $3 + 0 = (3)(1)$, or $3 = 3$. Therefore, the frame is statically determinate.

Note: When using equation 3.3 for classifying a frame, the frame must be disassembled at its joints to correctly determine the number of members.

Fig. 3.7. Frame.

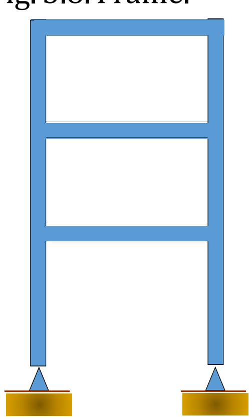


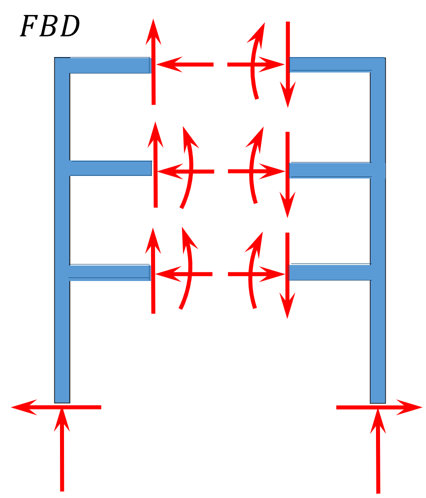
Solution

Using equation 3.3, $r = 6$, $m = 3$, $c = 1$, $j = 4$. Applying the equation leads to $3(3) + 6 > 3(4) + 1$, or $15 > 13$. Therefore, the frame is statically indeterminate to the 2°.

Using equation 3.4, $r = 6$, $m = 2$, $F_i = 2$. Applying the equation leads to$6 + 2 > 3(2)$, or $8 > 6$. Therefore, the frame is statically indeterminate to the 2°.

Fig. 3.8. Frame.





Solution

Using equation 3.3, $r = 4$, $m = 9$, $c = 0$, $j = 8$. Applying the equation leads to $3(9) + 4 > 3(8) + 0$, or $31 > 24$. Therefore, the frame is statically indeterminate to the 7°.

Using equation 3.4, $r = 4$, $m = 1, F_i = 9$. Applying the equation leads to $4 + 9 > (3)(2)$, or $13 > 6$. Therefore, the frame is statically indeterminate to the 7°.

Note: When using equation 3.4 to classify a frame with a closed loop, as given here, the loop has to be cut open by the method of section, and the internal reactions in the cut section should be considered in the analysis.

3.4 Computation of Support Reactions for Planar Structures

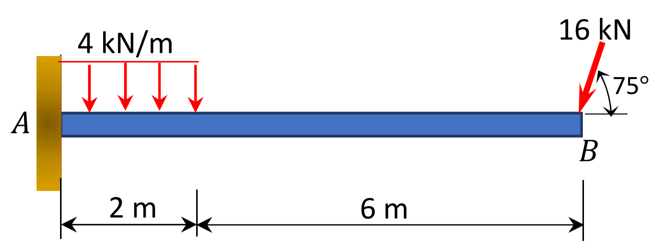
The support reactions for statically determinate and stable structures on a plane are determined by using the equations of equilibrium. The procedure for computation is outlined below.

Procedure for Computation of Support Reactions

- Sketch a free-body diagram of the structure, identifying all the unknown reactions using an arrow diagram.
- Check the stability and determinacy of the structure using equation 3.3 or 3.4. If the structure is classified as determinate, proceed with the analysis.
- Determine the unknown reactions by applying the three equations of equilibrium. If a computed reaction results in a negative answer, the initially assumed direction of the unknown reaction, as indicated by the arrow head on the free-body diagram, is wrong and should be corrected to show the opposite direction. Once the correction is made, the magnitude of the force should be indicated as a positive number in the corrected arrow head on the free-body diagram

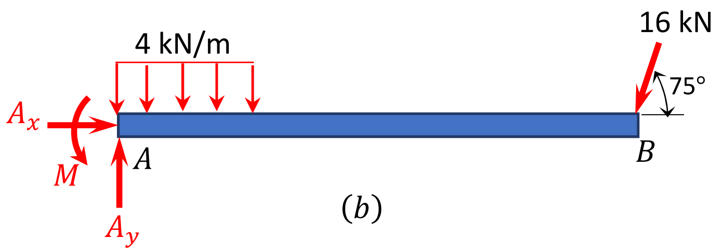
Example 3.3

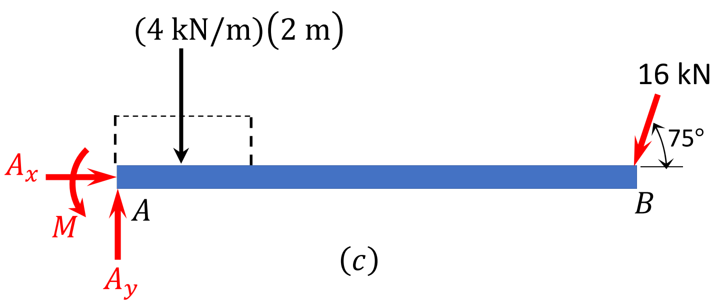
A cantilever beam is subjected to a uniformly distributed load and an inclined concentrated load, as shown in figure 3.9a. Determine the reactions at support *A*.



(a)

Fig. 3.9. Beam





Solution

Free-body diagram. The free-body diagram of the entire beam is shown in Figure 3.9b. The support reactions, as indicated in the free-body diagram, are A_y, A_x, and M.

Computation of reactions. Prior to the computation of the support reactions, the distributed loading should be replaced by a single resultant force, and the inclined loading resolved to the vertical and horizontal components. The magnitude of the resultant force is equal to the area under the rectangular loading, and it acts through the centroid of the rectangle. As seen in Figure 3.9c, $P = [(4\text{ kN/m})(2\text{ m})]$, and its location is at the centroid of the rectangle loading $= \left[\left(\frac{1}{2}\right)(2\text{ m})\right]$. Applying the equations of static equilibrium provides the following:

$$\curvearrowleft + \sum M_A = 0$$

$$-(16\sin 75°)(8) - (4 \times 2)(1) + M_A = 0$$

$$M_A = 131.64\text{ kN.m} \qquad\qquad M_A = 131.64\text{ kN.m} \curvearrowleft$$

$$\uparrow + \sum F_y = 0$$

$$A_y - 16 \sin 75° - (4 \times 2) = 0$$

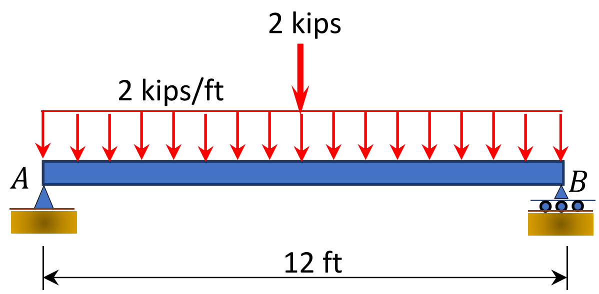
$$A_y = 23.45 \text{ kN} \qquad\qquad A_y = 23.45 \text{ kN} \uparrow$$

$$\rightarrow + \sum F_x = 0$$

$$A_x = 0 \qquad\qquad A_x = 0$$

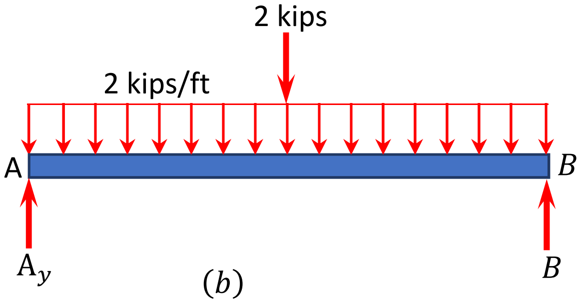
Example 3.4

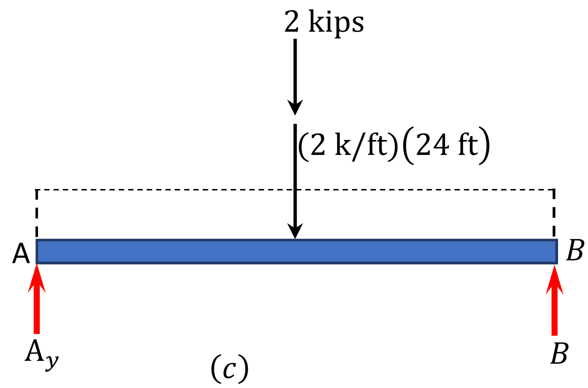
A 12ft-long simple beam carries a uniformly distributed load of 2 kips/ft over its entire span and a concentrated load of 8 kips at its midspan, as shown in Figure 3.10a. Determine the reactions at the supports A and B of the beam.



(a)

Fig. 3.10. Simple beam.





Solution

Free-body diagram. The free-body diagram of the entire beam is shown in Figure 3.10b.

Computation of reactions. The distributed loading is first replaced with a single resultant force, as seen in Figure 3.10c. The magnitude of the resultant force is equal to the area of the rectangular loading (distributed force). Thus, $P = [(2 \text{ k/ft})(12 \text{ ft})]$, and its location is at the centroid of the rectangular loading = $\left[\left(\frac{1}{2}\right)(12\text{ft})\right]$. Since there is a symmetry in loading in this example, the reactions at both ends of the beam are equal, and they could be determined using the equations of static equilibrium and the principle of superposition, as follows:

$+\uparrow \sum F_y = 0$

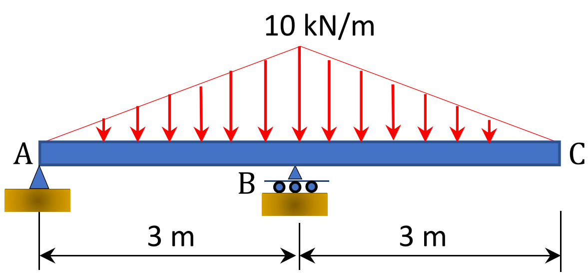
$A_y = B_y = \left(\frac{2\times 12}{2}\right) + \frac{2}{2} = 13 \text{ kips}$ $\qquad A_y = B_y = 13 \text{ kips} \uparrow$

$+\rightarrow \sum \mathrm{F_x} = 0$

$\mathrm{A_x} = 0$ $\qquad \mathrm{A_x} = 0$

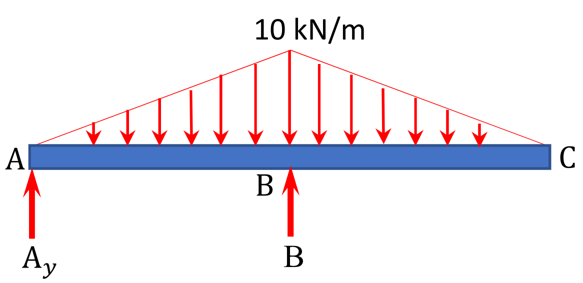
Example 3.5

A beam with an overhang is subjected to a varying load, as shown in Figure 3.11a. Determine the reactions at supports A and B.

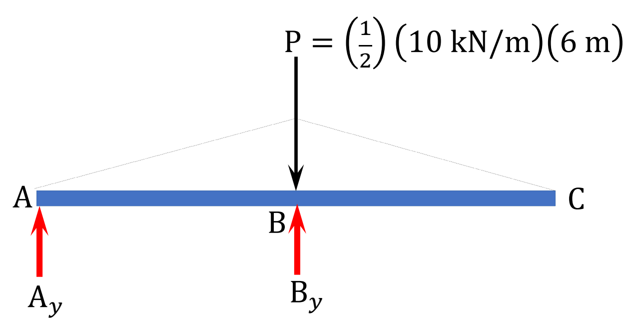


(a)

Fig. 3.11. Beam with an overhang.



(b)



(c)

Solution

Free-body diagram. The free-body diagram of the entire beam is shown in Figure 3.11b.

Computation of reactions. Observe that the distributed loading in the beam is triangular. The distributed load is first replaced with a single resultant force, as shown in Figure 3.11c. The magnitude of the single resultant force is equal to the area under the triangular loading. Thus, $P = \left(\frac{1}{2}\right)(6\text{ m})(10\text{ kN/m})$, and its centroid is at the center of the loading (6m). Applying the equations of equilibrium provides the following:

$$\curvearrowleft + \sum M_A = 0$$

$$-\left(\frac{1}{2}\right)(10)(6)(3) + 3\text{B} = 0$$

$$B_y = 30\text{ kN} \qquad\qquad B_y = 30\text{ kN} \uparrow$$

$$\uparrow + \sum F_y = 0$$

$$30 + A_y - \left(\frac{1}{2}\right)(6)(10) = 0 \qquad\qquad A_y = 0$$

$$\rightarrow + \sum \text{F}_\text{x} = 0$$

$$\text{A}_\text{x} = 0 \qquad\qquad \text{A}_\text{x} = 0$$

Example 3.6

A beam with overhanging ends supports three concentrated loads of 12 kips, 14 kips, and 16 kips and a moment of 100 kips.ft, as shown in Figure 3.12a. Determine the reactions at supports *A* and *B*.

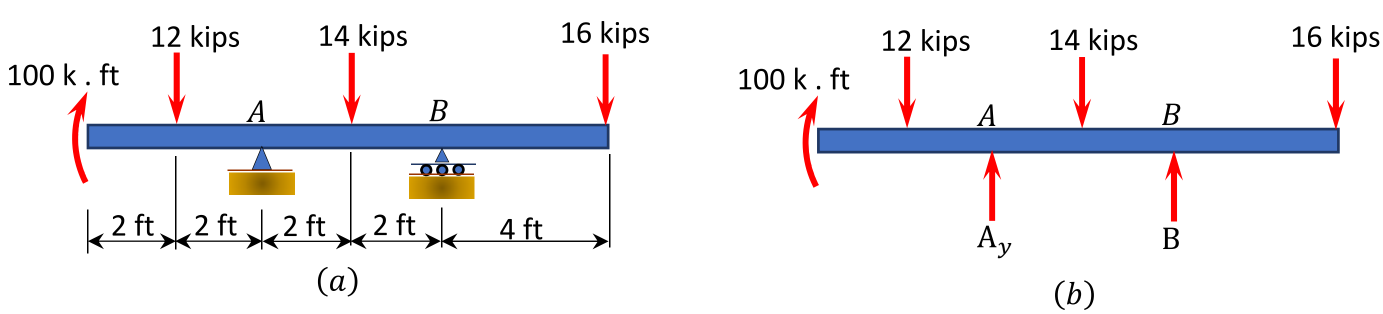


Fig. 3.12. Beam with overhanging ends.

Solution

Free-body diagram. The free-body diagram of the beam is shown in Figure 3.12b.

Computation of reactions. Applying the equations of equilibrium provides the following:

$$+\circlearrowleft \sum M_A = 0$$

$$-100 + 12(2) - 14(2) - 16(8) + 4B_y = 0$$

$B_y = 58 \text{ kips}$ $\qquad\qquad$ $B_y = 58 \text{ kips} \uparrow$

$$+\uparrow \sum F_y = 0$$

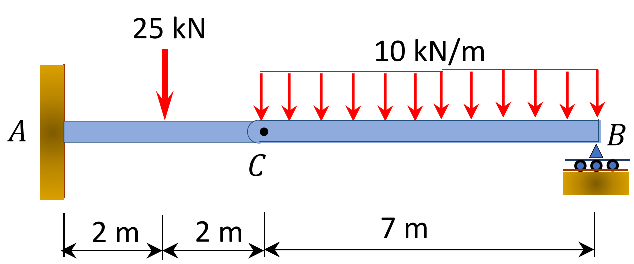
$58 + A_y - 12 - 14 - 16 = 0$ $\qquad\qquad$ $A_y = 16 \text{ kips} \uparrow$

$$+\rightarrow \sum F_x = 0$$

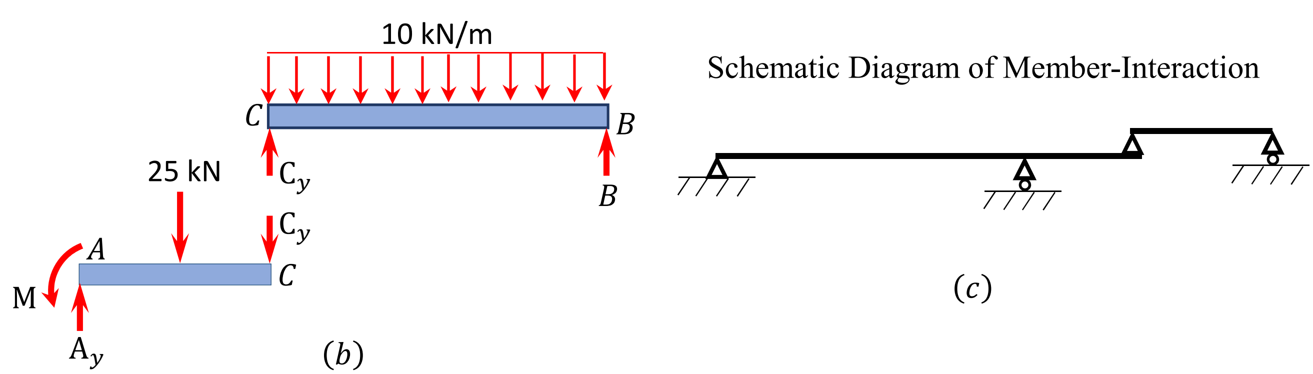
$A_x = 0$ $\qquad\qquad$ $A_x = 0$

Example 3.7

A compound beam is subjected to the loads shown in Figure 3.13a. Find the support reactions at A and B of the beam.



(a) Fig. 3.13. Compound beam.



(b) $\qquad$ (c)

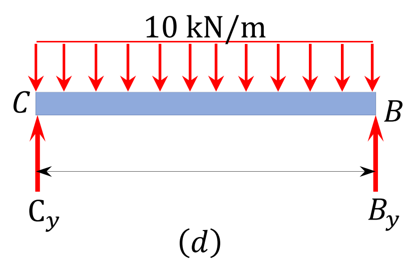
Solution

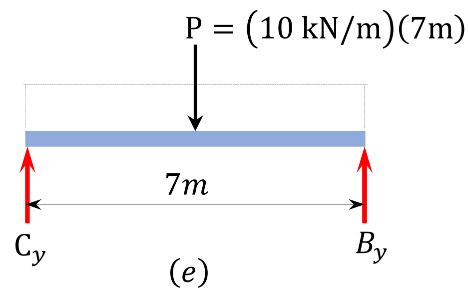
Free-body diagram. The free-body diagram of the entire beam is shown in Figure 3.13b.

Identification of primary and complimentary structures. For correct analysis of a compound structure, the primary and the complimentary parts of the structure should be identified for proper understanding of their interaction. The interaction of these parts are shown in Figure 3.13c. The primary structure is the part of the compound structure that can sustain the applied external load without the assistance of the complimentary structure. On the other hand, the complimentary structure is the part of the compound structure that depends on the primary structure to support the applied external load. For the given structure, part *AC* is the primary structure, while part *CB* is the complimentary structure.

Computation of reactions. The analysis of a compound structure must always begin with the analysis of the complimentary structure, as the complimentary structure is supported by the primary structure. Using the equations of equilibrium, the support reactions of the beam are determined as follows:

Analysis of the complimentary structure *CB*.



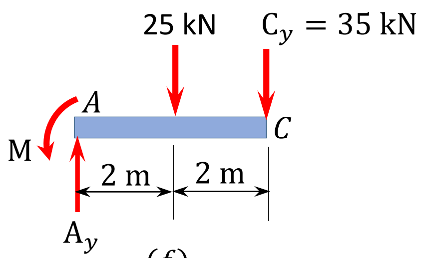


Computation of support reaction. The isolated free-body diagram of the complimentary structure is shown in Figure 3.13c. First, the distributed loading is replaced by a single resultant force (*P*), which is equal to the area of the rectangular loading, as shown in Figure 3.13d and Figure 3.13e. Applying the equations of equilibrium, and noting that due to symmetry in loading, the support reactions at point *C* and *B* are equal in magnitude, provides the following:

$+\uparrow \sum F_y = 0$

$B_y = C_y = \frac{10(7)}{2} = 35 \text{ kN}$ $\qquad$ $B_y = C_y = \frac{10(7)}{2} = 35 \text{ k} \uparrow$

Analysis of the primary structure *AC*.



(f)

Computation of support reaction. Note that prior to the computation of the reactions, the reaction at point *C* in the complimentary structure is applied to the primary structure as a load. The magnitude of the applied load is the same as that of the complimentary structure, but it is opposite in direction. Applying the equations of equilibrium suggests the following:

$$+\circlearrowleft \sum M_A = 0$$

$$-25(2) - 35(4) + M_A = 0$$

$$M_A = 190 \text{ kN.m} \qquad M_A = 190 \text{ kN.m} \circlearrowleft$$

$$+\uparrow \sum F_y = 0$$

$$A_y - 25 - 35 = 0$$

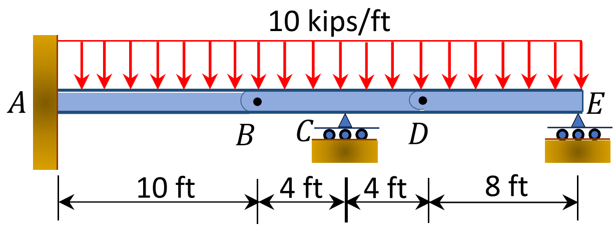
$$A_y = 60 \text{ kN} \qquad A_y = 60 \text{ kN} \uparrow$$

$$+\rightarrow \sum F_x = 0$$

$$A_x = 0 \qquad A_x = 0$$

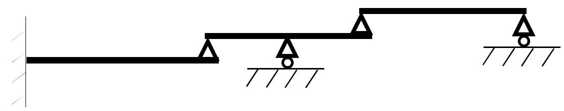
Example 3.8

Find the reactions at supports A, C, and E of the compound beam carrying a uniformly distributed load of 10 kips/ft over its entire length as shown in figure 3.14a.



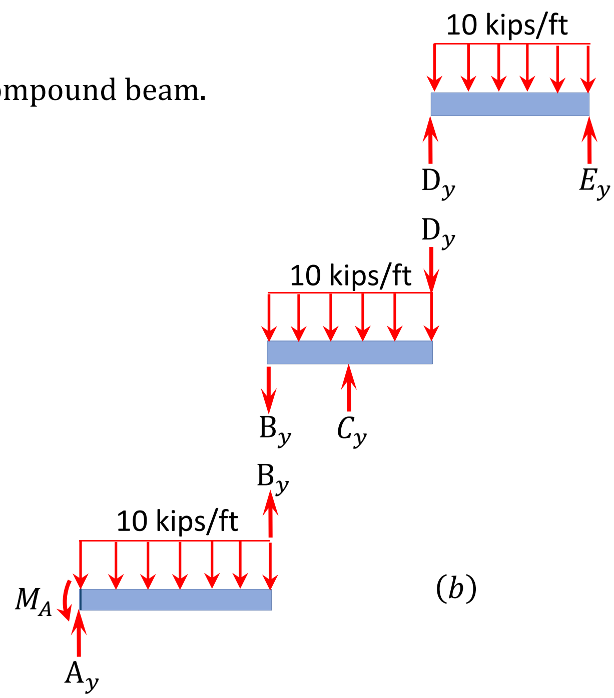
(a)

Schematic Diagram of Member-Interaction



(c)

Fig. 3.14. Compound beam.



(b)

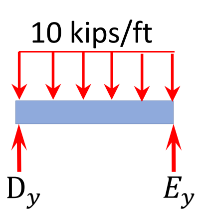
Solution

Free-body diagram. The free-body diagram of the entire beam is shown in Figure 3.14b.

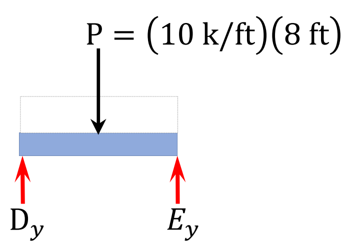
Identification of primary and complimentary structures. The interaction diagram for the given structure is shown in Figure 3.14c. *AB* is the primary structure, while *BD* and *DE* are the complimentary structures.

Computation of reactions.

Analysis of complimentary structure *DE*.



(c)



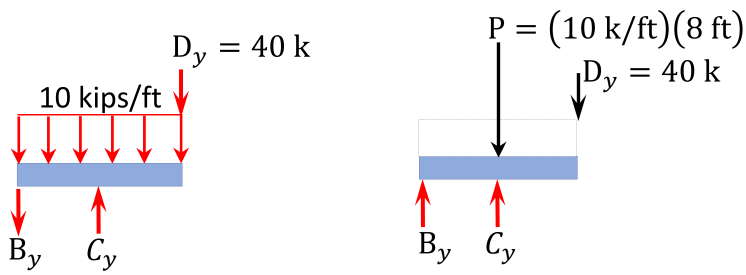
(d)

Computation of support reaction. The isolated free-body diagram is shown in Figure 3.14c. First, the distributed loading is replaced by a single resultant force (*P*) equal the area of rectangular loading, as shown in Figure 3.14d. Applying the equations of equilibrium, and noting that due to symmetry in loading, the support reactions at point *D* and *E* are equal in magnitude, suggests the following:

$+\uparrow \sum F_y = 0$

$D_y = E_y = \frac{10(8)}{2} = 40 \text{ kips}$ $\qquad E_y = 40 \text{ kips} \uparrow$

Analysis of complimentary structure *BD*.



(*e*) (*f*)

Computation of support reaction. The isolated free-body diagram is shown in Figure 3.14e. First, the distributed loading is replaced by a single resultant force (*P*) equal to the area of the rectangular loading, as shown in Figure 3.14f. The load from the complimentary structure is applied at point *D*. Applying the equations of equilibrium suggests the following:

$+\curvearrowleft \sum M_B = 0$

$-10(8)\left(\frac{8}{2}\right) - 40(8) + 4C_y = 0$

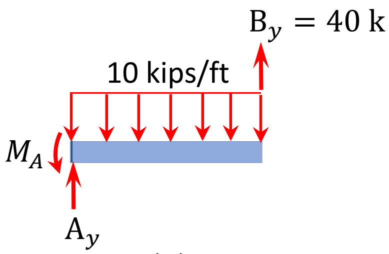
$C_y = 160 \text{ kips}$ $\qquad C_y = 160 \text{ kips} \uparrow$

$+\uparrow \sum F_y = 0$

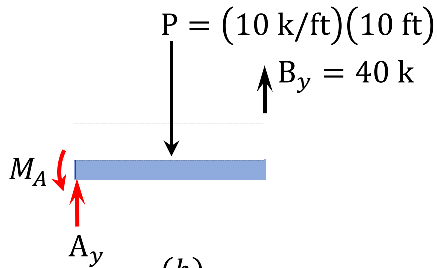
$160 - B_y - 10(8) - 40 = 0$

$B_y = 40 \text{ kips}$

Analysis of primary structure *AB*.



(g)



(h)

Computation of support reaction. Note that prior to the computation of the reactions, the uniform load is replaced by a single resultant force, and the reaction at point *B* in the complimentary structure is applied to the primary structure as a load. Applying the equilibrium requirement yields the following:

$$+\curvearrowleft \sum M_A = 0$$

$$\mathrm{M} - 10(10)\left(\frac{10}{2}\right) + 40(10) = 0$$

$$M_A = 100 \text{ kips. ft} \qquad M_A = 100 \text{ kips. ft} \curvearrowleft$$

$$+\uparrow \sum F_y = 0$$

$$A_y - 10(10) + 40 = 0$$

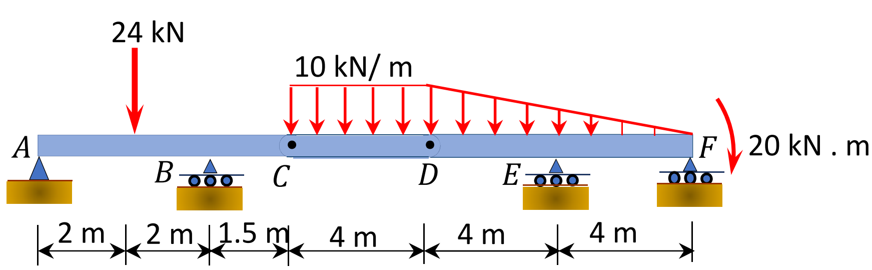
$$A_y = 60 \text{ kips} \qquad A_y = 60 \text{ kips} \uparrow$$

$$+\rightarrow \sum \mathrm{F_x} = 0$$

$$A_x = 0 \qquad A_x = 0$$

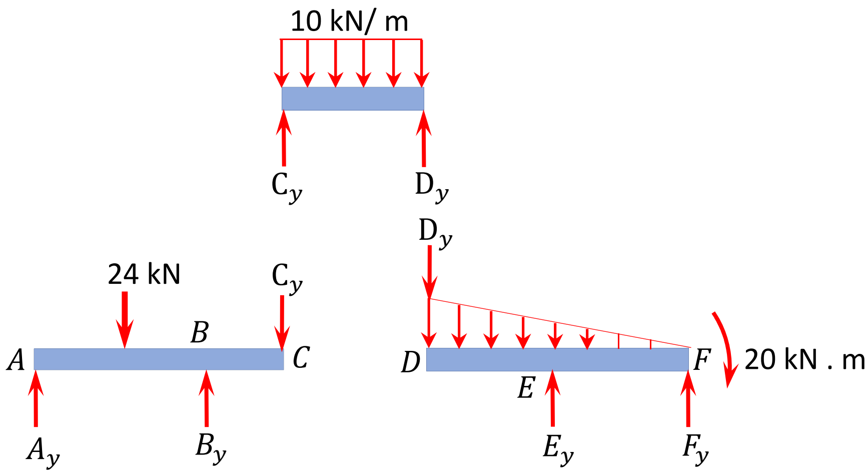
Example 3.9

Find the reactions at supports *A*, *B*, *E*, and *F* of the loaded compound beam, as shown in Figure 3.15a.



(a)

Fig. 3.15. Compound beam.



(b)

Schematic Diagram of Member-Interaction

(c)

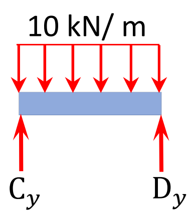
Solution

Free-body diagram. The free-body diagram of the entire beam is shown in Figure 3.15b.

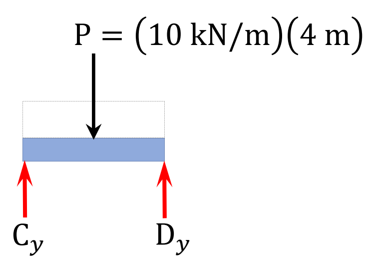
Identification of primary and complimentary structure. The interaction diagram for the given structure is shown in Figure 3.15c. *CD* is the complimentary structure, while *AC* and *DF* are the primary structures.

Computation of reactions.

Analysis of complimentary structure *CD.*



(c)



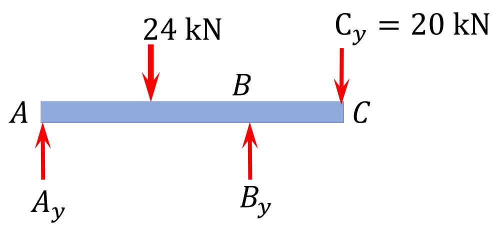
(d)

Computation of support reaction. The isolated free-body diagram is shown in Figure 3.15c. First, the distributed loading is replaced by a single resultant force (P), which is equal to the area of the rectangular loading, as shown in Figure 3.15d. Applying the equations of equilibrium, and noting that due to symmetry in loading, the support reactions at point C and D are equal in magnitude, suggests the following:

$+\uparrow \sum F_y = 0$

$C_y = D_y = \frac{10(4)}{2} = 20 \text{ kN}$

Analysis of primary structure *AC*.



(e)

Computation of support reaction. Note that the reaction at C of the complimentary structure is applied as a downward force of the same magnitude at the same point on the primary structure. Applying the equation of equilibrium suggests the following:

$+\circlearrowleft \sum M_A = 0$

$-24(2) - 20(5.5) + 4B_y = 0$

$B_y = 39.5 \text{ kN}$ $\qquad\qquad$ $B_y = 39.5 \text{ kN} \uparrow$

$+\uparrow \sum F_y = 0$

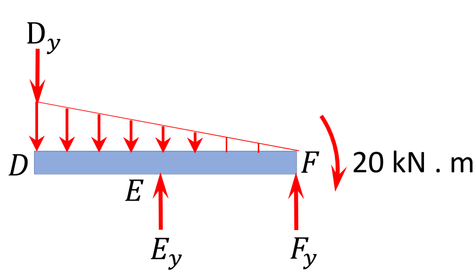
$A_y + 39.5 - 24 - 20 = 0$

$A_y = 4.5 \text{ kN}$ $\qquad\qquad$ $A_y = 4.5 \text{ kN} \uparrow$

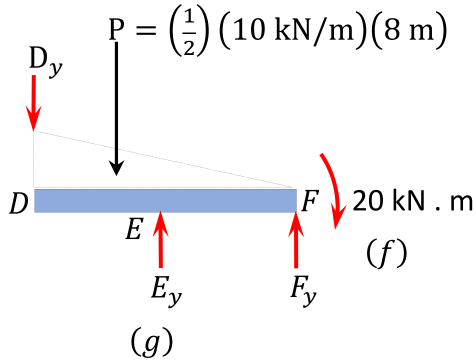
$+\rightarrow \sum F_x = 0$

$A_x = 0$ $\qquad\qquad$ $A_x = 0$

Analysis of primary structure *DF*.



(f)



Computation of support reaction. The isolated free-body diagram is shown in Figure 3.15f. First, the distributed loading is replaced by a single resultant force (P) equal to the area of the triangular loading, as shown in Figure 3.15g. Applying the equations of equilibrium, and noting that the support reaction at point D of the complimentary structure is applied as a load on the primary structure, suggests the following:

$$+\circlearrowleft \sum M_F = 0$$
$$-20 + \left(\tfrac{1}{2} \times 8 \times 10\right)\left(\tfrac{2}{3} \times 8\right) + 20(8) - 4E_y = 0$$
$$E_y = 88.33 \text{ kN} \qquad\qquad E_y = 88.33\text{kN} \uparrow$$

$$+\uparrow \sum F_y = 0$$
$$F_y + 88.33 - \left(\tfrac{1}{2} \times 8 \times 10\right) - 20 = 0$$
$$F_y = 28.33 \text{ kN} \qquad\qquad F_y = 28.33 \text{ kN} \uparrow$$

Example 3.10

Determine the reactions at supports A and D of the frame shown in Figure 3.16a.

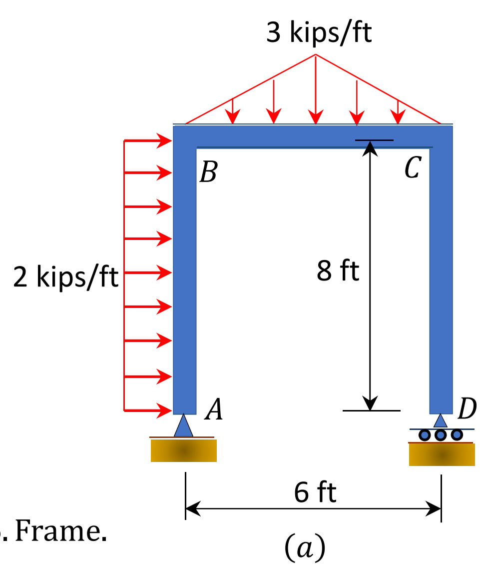
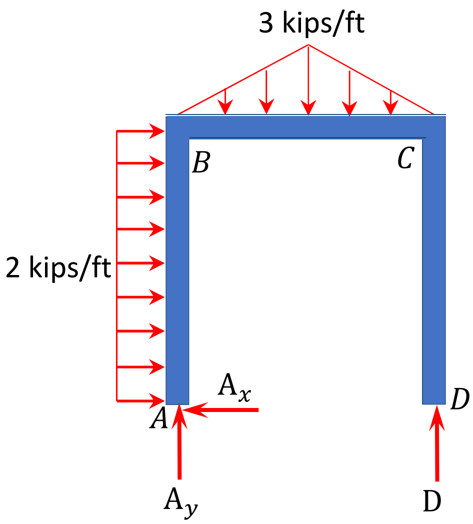
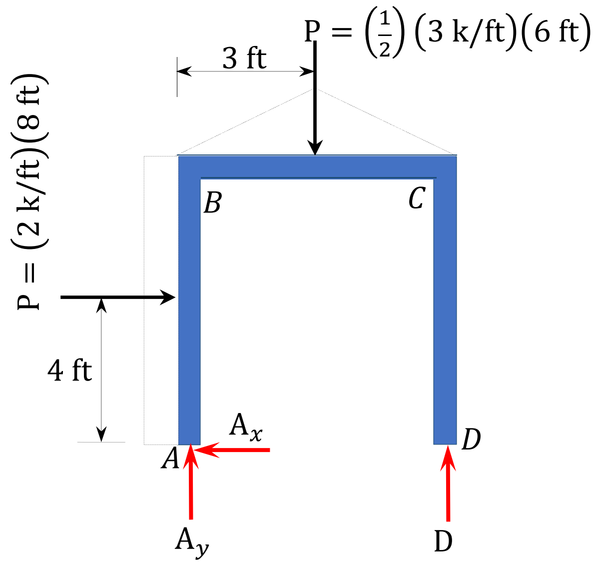


Fig. 3.16. Frame.



(b)



(c)

Solution

Free-body diagram. The free-body diagram of the entire beam is shown in Figure 3.16b.

Computation of reactions. The distributed loads in column *AB* and beam *BC* are first replaced by single resultant forces determined as the area of their respective shade of loading, as shown in Figure 3.16c. Applying the conditions of equilibrium suggests the following:

$$+\circlearrowleft \sum M_A = 0$$

$$D_y(6) - \left(\tfrac{1}{2}\right)(6)(3)(3) - (2)(8)(4) = 0$$

$D_y = 15.7$ kips $\qquad\qquad D_y = 15.7$ kips $\uparrow$

$$+\uparrow \sum F_y = 0$$

$$A_y + 15.17 - 3(6) = 0$$

$A_y = 2.830$ kips $\qquad\qquad A_y = 2.830$ kips $\uparrow$

$$+\rightarrow \sum F_x = 0$$

$$-A_x + (2 \times 8) = 0$$

$A_x = 16$ kips $\qquad\qquad A_x = 16$ kips $\leftarrow$

Example 3.11

A rigid frame is loaded as shown in Figure 3.17a. Determine the reactions at support *D*.

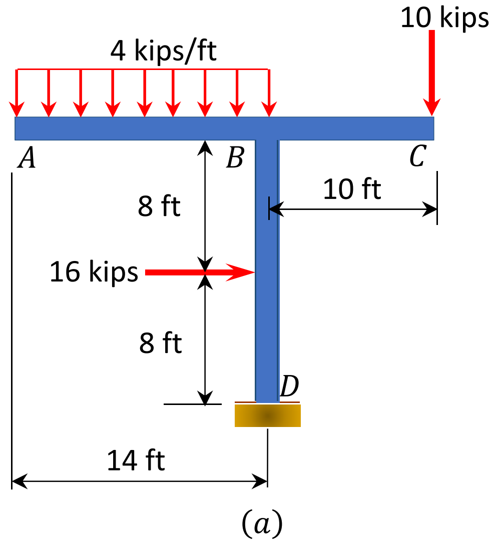
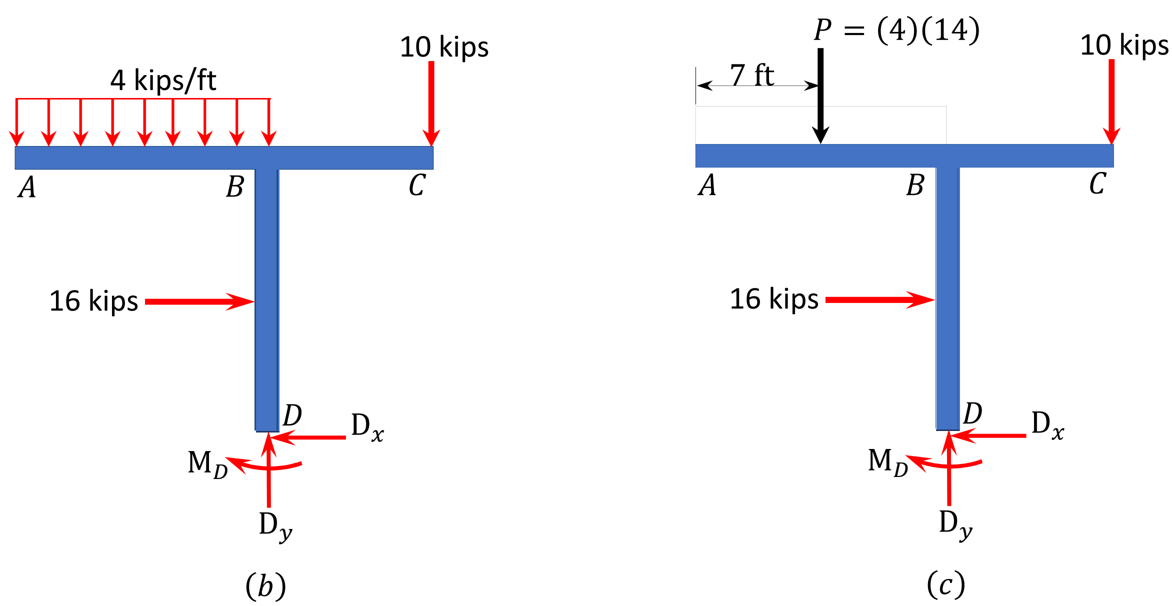


Fig. 3.17. Rigid frame.



Solution

Free-body diagram. The free-body diagram of the entire beam is shown in Figure 3.17b.

Computation of reactions. The distributed load in portion *AB* of the frame is first replaced with a single resultant force, as shown in Figure 3.17c. Applying the equations of equilibrium suggests the following:

$$+\curvearrowleft \sum M_D = 0$$

$$-M_D - 16(8) + (4 \times 14)\left(\frac{14}{2}\right) - 10(10) = 0$$

$M_D = 164 \text{ kips.ft}$ $\qquad\qquad$ $M_D = 164 \text{ kips.ft} \curvearrowright \text{A}$

$$+\uparrow \sum F_y = 0$$

$$D_y - 4(14) - 10 = 0$$

$D_y = 66 \text{ kips}$ $\qquad\qquad$ $D_y = 66 \text{ kips} \uparrow$

$$+\rightarrow \sum \mathrm{F_x} = 0$$

$$-D_x + 16 = 0$$

$D_x = 16 \text{ kips}$ $\qquad\qquad$ $D_x = 16 \text{ kips} \leftarrow$

Example 3.12

Find the reactions at supports *E* and *F* of the frame shown in Figure 3.18a.

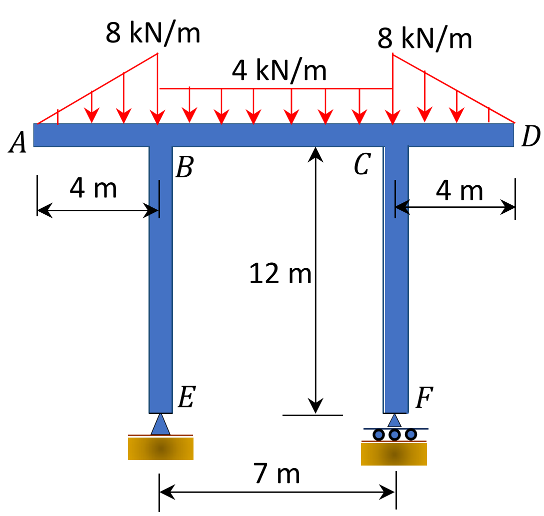
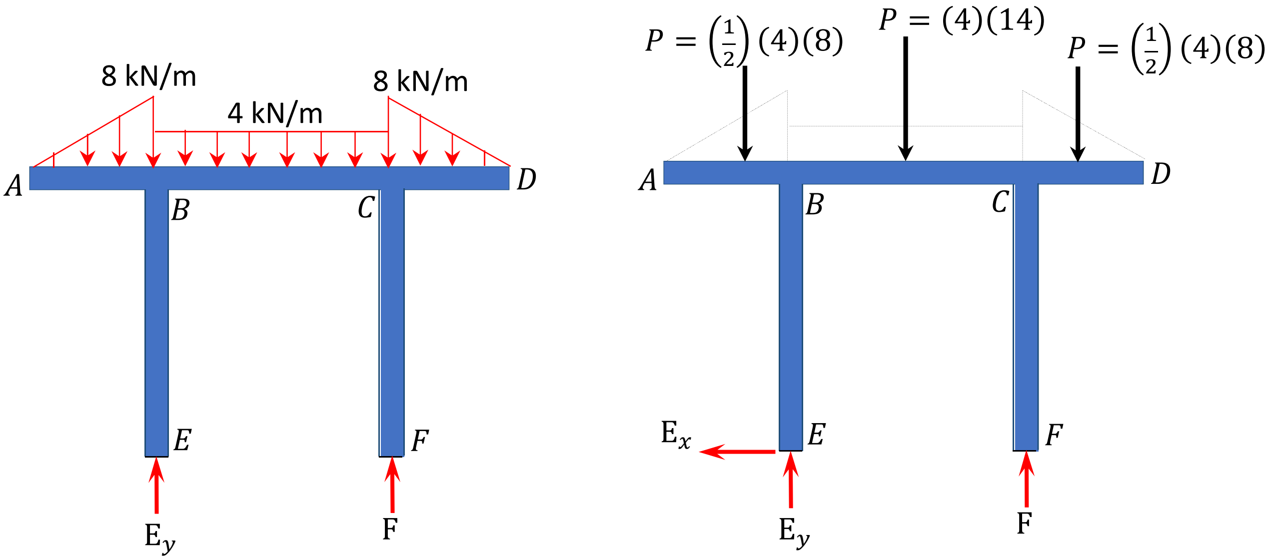


Fig. 3.18. Frame. $\qquad (a)$



(b) (c)

Solution

Free-body diagram. The free-body diagram of the frame is shown in Figure 3.18b.

Computation of reactions. The distributed loads are first replaced with single resultant forces, as shown in Figure 3.18c. Applying the equations of static equilibrium suggests the following:

$$+\curvearrowleft \sum M_E = 0$$

$$\left(\tfrac{1}{2}\times 4\times 8\right)\left(\tfrac{1}{3}\times 4\right) - (4\times 7)\left(\tfrac{7}{2}\right) - \left(\tfrac{1}{2}\times 4\times 8\right)\left(\tfrac{7}{2}+\tfrac{1}{3}\times 4\right) + 7F_y = 0$$

$F_y = 22\text{ kN}$ $\qquad$ $F_y = 22\text{ kN}\uparrow$

$$+\uparrow \sum F_y = 0$$

$$E_y + 22 - 2\left(\tfrac{1}{2}\times 4\times 8\right) - 4(7) = 0$$

$E_y = 38\text{ kN}$ $\qquad$ $E_y = 38\text{ kN}\uparrow$

$$+\rightarrow \sum \mathrm{F_x} = 0$$

$E_x = 0$ $\qquad$ $E_x = 0$

Example 3.13

Determine the reactions at support *A* of the rigid frame shown in Figure 3.19a.

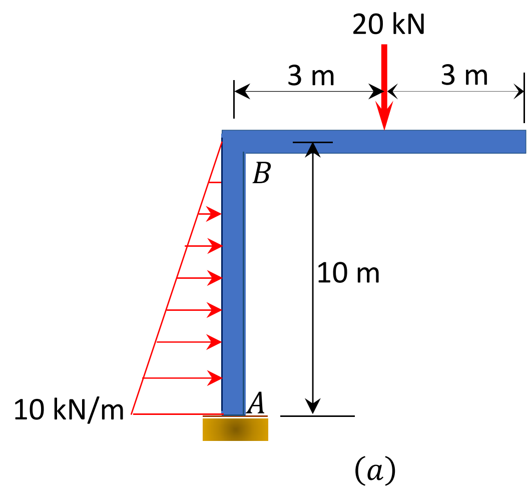
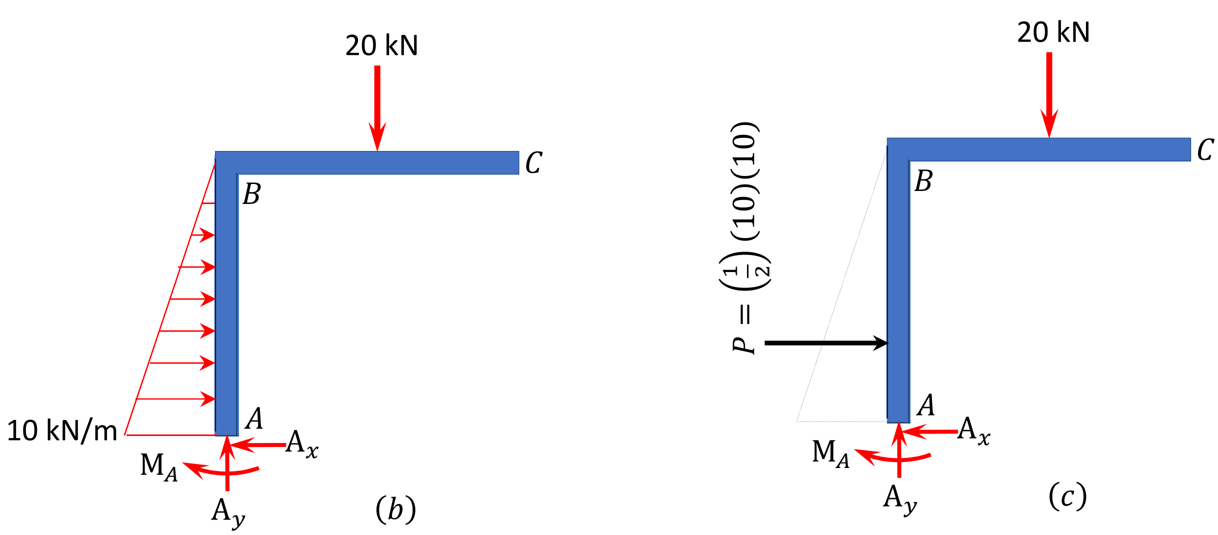


Fig. 3.19. Rigid frame.



Solution

Free-body diagram. The free-body diagram of the frame is shown in Figure 3.19b.

Computation of reactions. The distributed load in column AB is first replaced with a single resultant force, as shown in Figure 3.19c. Applying the equations of static equilibrium suggests the following:

$$+\circlearrowleft \sum M_A = 0$$

$$-M_A - 20(3) - \left(\frac{1}{2} \times 10 \times 10\right)\left(\frac{1}{3} \times 10\right) = 0$$

$M_A = -226.67 \text{ kN.m}$ $\qquad$ $M_A = 226.67 \text{ kN.m} \circlearrowleft$

$$+\uparrow \sum F_y = 0$$

$A_y - 20 = 0$

$A_y = 20 \text{ kN}$ $\qquad$ $A_y = 20 \text{ kN} \uparrow$

$+\rightarrow \sum F_x = 0$

$-A_x + \left(\frac{1}{2} \times 10 \times 10\right) = 0$

$A_x = 50 \text{ kN}$ $\qquad$ $A_x = 50 \text{ kN} \leftarrow$

Example 3.14

Determine the reactions at supports A and E of the frame hinged at C, as shown in Figure 3.20a.

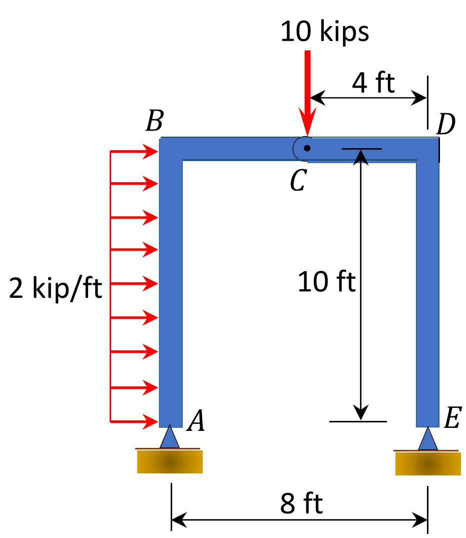
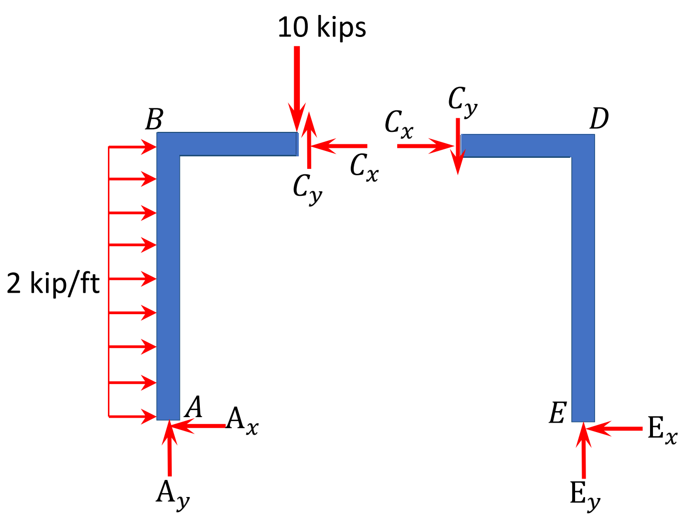
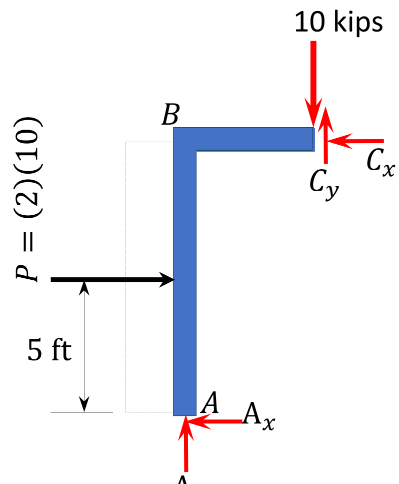


Fig. 3.20. Frame.

(a)



(b)



(c)

Solution

Free-body diagram. The free-body diagram of the frame is shown in Figure 3.20b.

Computation of reactions. The reactions in a compound frame are computed considering the free-body diagrams of both the entire frame and part of the frame. Prior to computation of the reactions, the distributed load in the column is replaced by a single resultant force. The vertical reactions at *E* and *A* and the horizontal reactions at *A* are found by applying the equations of static equilibrium and considering the free-body diagram of the entire frame. The horizontal reaction at *E* is found by considering part *CDE* of the free-body diagram.

$$+\curvearrowleft \sum M_A = 0$$

$$8E_y - (2 \times 10)\left(\frac{10}{2}\right) - 10(4) = 0$$

$E_y = 17.5 \text{ kips}$ $\qquad\qquad$ $E_y = 17.5 \text{ kips} \uparrow$

$$+\uparrow \sum F_y = 0$$

$$A_y + 17.5 - 10 = 0$$

$A_y = -7.5 \text{ kips}$ $\qquad\qquad$ $A_y = 7.5 \text{ kips} \downarrow$

The negative sign implies that the originally assumed direction of A_y was not correct. Therefore, A_y acts downward instead of upward as was initially assumed. This should be corrected in the subsequent analysis.

To determine E_x, consider the moment of forces in member *CDE* about the hinge.

$$\curvearrowleft + \sum M_C = 0$$

$17.5(4) - 10E_x = 0$
$E_x = 7$ kips $E_x = 7\text{kips} \leftarrow$

$+\rightarrow \sum F_x = 0$
$-A_x - 7 + 2 \times 10 = 0$
$A_x = 13$ kips $A_x = 13 \text{ kips} \leftarrow$

Example 3.15

Find the reactions at support *A* and *B* of the loaded frame in Figure 3.21a. The frame is hinged at *D*.

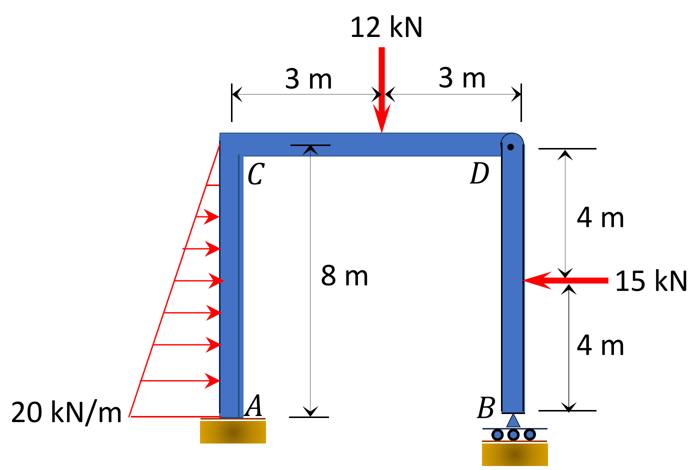
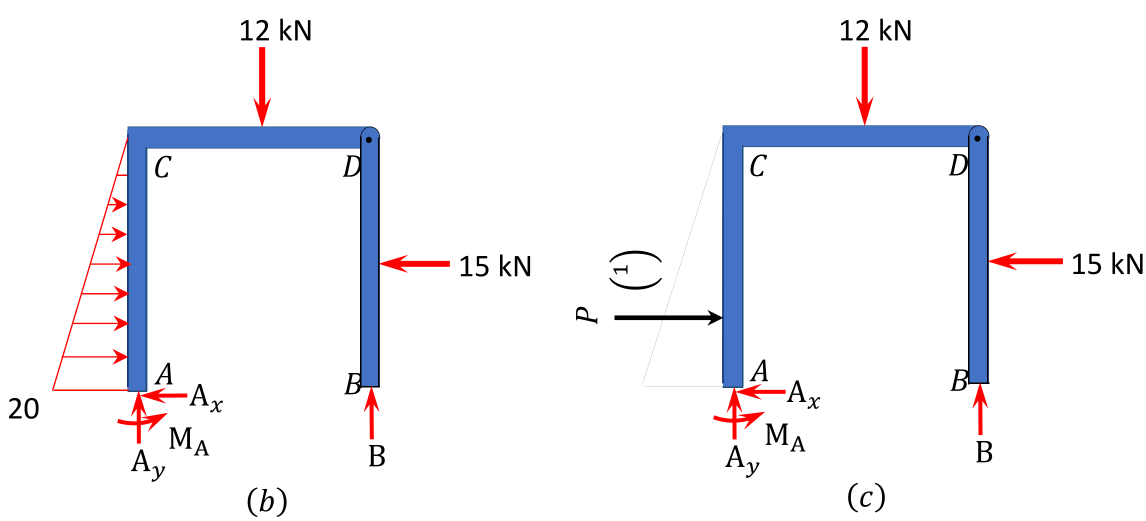


Fig. 3.21. Loaded frame. (a)



Solution

Free-body diagram. The free-body diagram of the frame is shown in Figure 3.21b.

Computation of reactions. The distributed load in column *AC* is first replaced with a single resultant force by finding the area of loading, as shown in Figure 3.21Figurec. The reaction at *B* is computed by taking the moment of the forces in part *DB* of the frame about the pin at *D*, and other reactions are determined by applying other conditions of equilibrium.

$+\circlearrowleft \sum M_D = 0$

$B_y(0) - 15(4) = 0$
$B_y = 0$
$+\circlearrowleft \sum M_A = 0$

$M_A + 6 \times 0 - \left(\frac{1}{2} \times 8 \times 20\right)\left(\frac{1}{3} \times 8\right) - 12(3) + 15(4) = 0$

$M_A = 189.33 \text{ kN.m}$ $\qquad M_A = 189.33 \text{ kN.m} \curvearrowright$

$+\uparrow \sum F_y = 0$

$A_y + 0 - 12 = 0$
$A_y = -12 \text{ kN}$ $\qquad A_y = 12 \downarrow$

The negative sign implies that the originally assumed direction of A_y was not correct. Therefore, A_y acts downward instead of upward as was initially assumed. This should be corrected in the subsequent analysis.

$+\rightarrow \sum F_x = 0$
$-A_x - 15 + \left(\frac{1}{2} \times 8 \times 20\right) = 0$
$A_x = 65 \text{ kN}$ $\qquad A_x = 65 \text{ kN} \rightarrow$

Chapter Summary

Conditions of static equilibrium: A structure is in a state of static equilibrium if the resultant of all the forces and moments acting on it is equal to zero. Mathematically, this is expressed as follows:

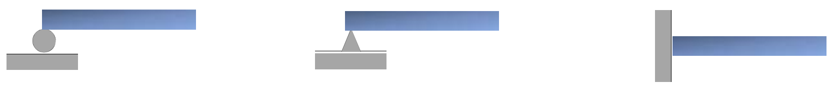
$$\sum F = 0 \qquad \sum M = 0$$

For a body in a plane, there are the following three equations of equilibrium:

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_o = 0$$

Types of support: Various symbolic representations are used to model different types of supports for structures. A roller is used to model a support that prevents a vertical movement of a structure but allows a horizontal translation and rotation. A pin is used to model a support that prevents horizontal and vertical movements but allows rotation. A fixed support models a support that prevents horizontal and vertical movements and rotation.

$V = 0$ $H \neq 0$ $\theta \neq 0$	$V = 0$ $H = 0$ $\theta \neq 0$	$V = 0$ $H = 0$ $\theta = 0$



Determinacy, indeterminacy, and stability of structures: A structure is determinate if the number of unknown reactions is equal to the number of static equilibrium. Thus, the equations of static equilibrium are enough for the determination of the supports for such a structure. On the other hand, a statically indeterminate structure is a structure that has the number of the unknown reactions in excess of the equations of equilibrium. For the analysis of an indeterminate structure additional equations are needed, and these equations can be obtained by considering the compatibility of the structure. Indeterminate structures are sometimes necessary when there is a need to reduce the sizes of members or to increase the stiffness of members. A stable structure is one which has support reactions that are not parallel or concurrent to one another. The formulation of stability and determinacy of beams and frames are as follows:

Beams and frames: $3m + r < 3j + C$ Structure is unstable

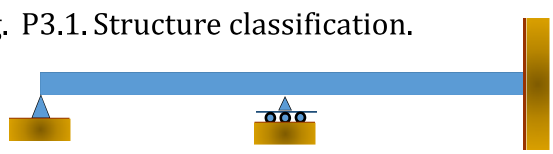
$3m + r = 3j + C$ Structure is determinate

$3m + r > 3j + C$ Structure is indeterminate

Practice Problems

3.1 Classify the structures shown in Figure P3.1a to Figure P3.1p as statically determinate or indeterminate, and statically stable or unstable. If indeterminate, state the degree of indeterminacy.

Fig. P3.1. Structure classification.

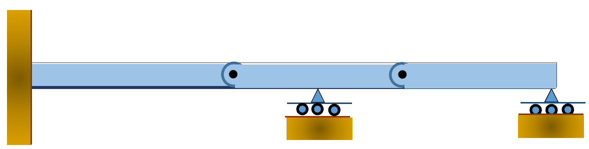


(a)

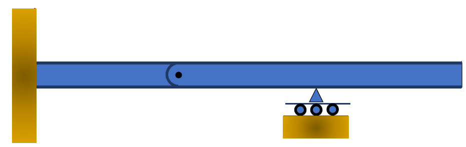
(b)

(c)

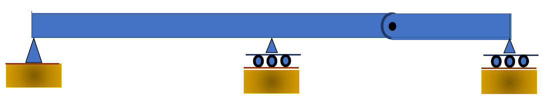
(d)



(e)

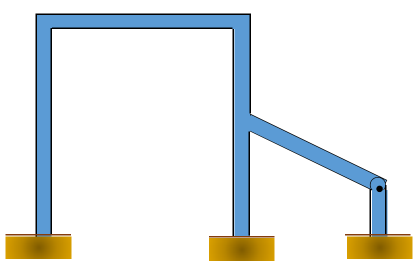


(f)

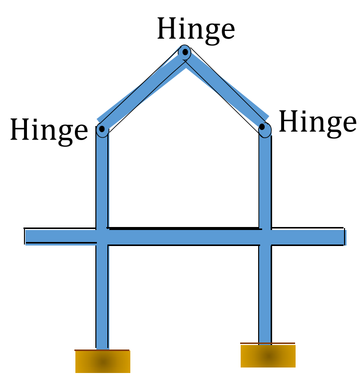


(g)

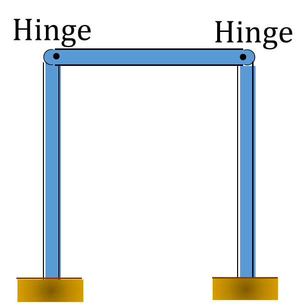
(h)



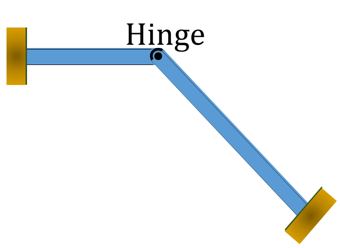
(i)



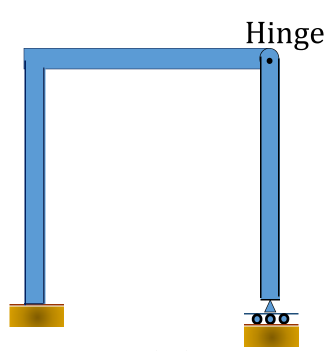
(j)



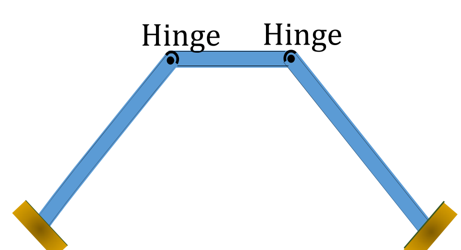
(k)



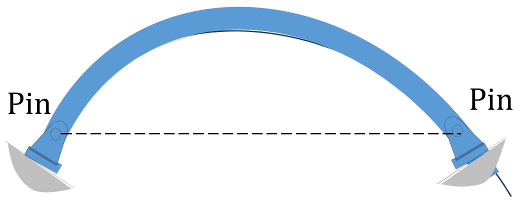
(l)



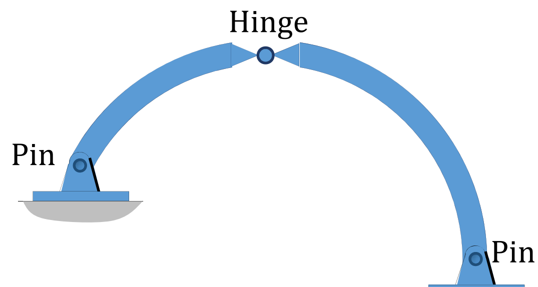
(m)



(n)



(o)



(p)

3.2. Determine the support reactions for the beams shown in Figure P3.2 through Figure P3.12.

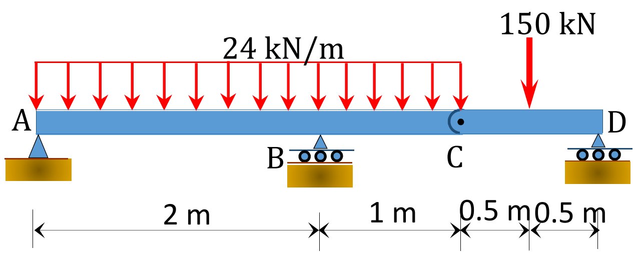


Fig. P3.2. Beam.

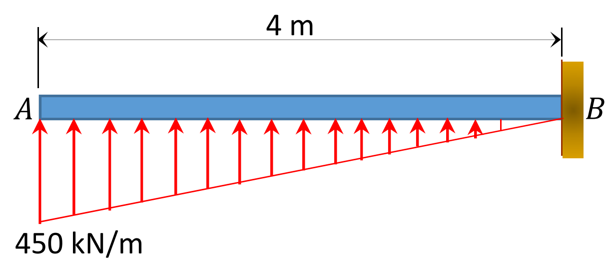


Fig. P3.3. Beam.

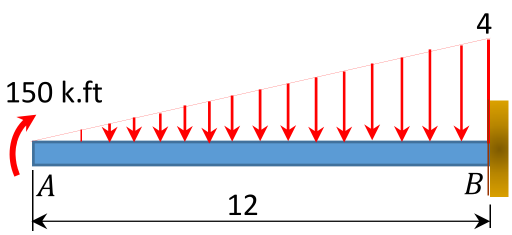


Fig. P3.4. Beam.

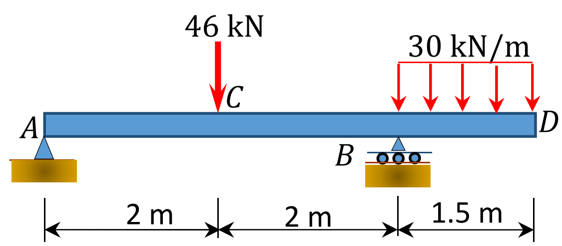


Fig. P3.5. Beam.

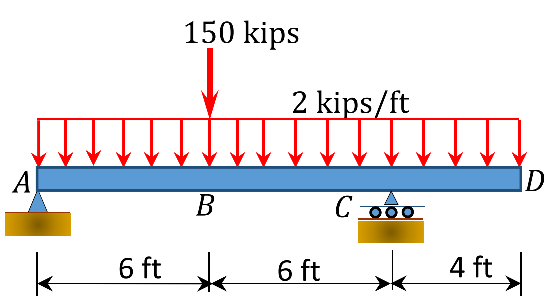


Fig. P3.6. Beam.

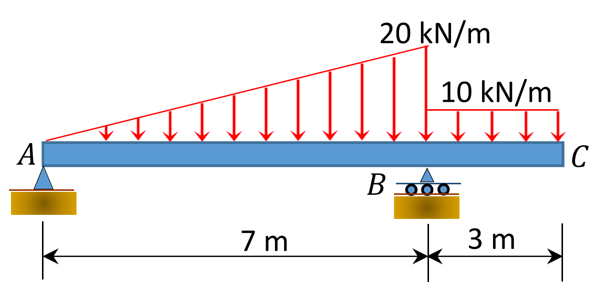


Fig. P3.7. Beam.

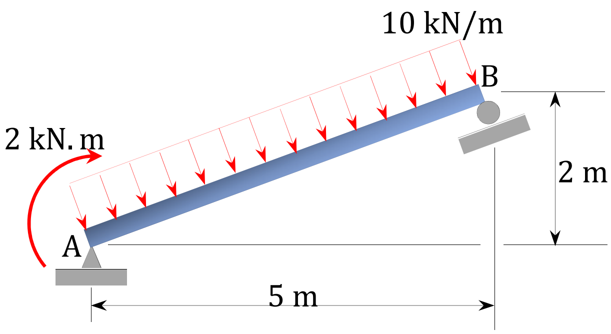


Fig. P3.8. Beam.

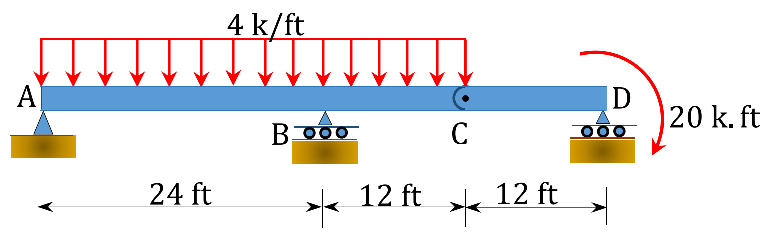


Fig. P3.9. Beam.

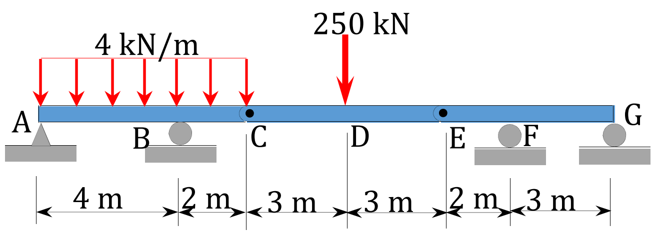


Fig. P3.10. Beam.

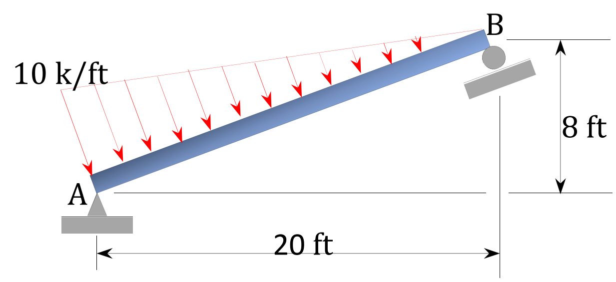


Fig. P3.11. Beam.

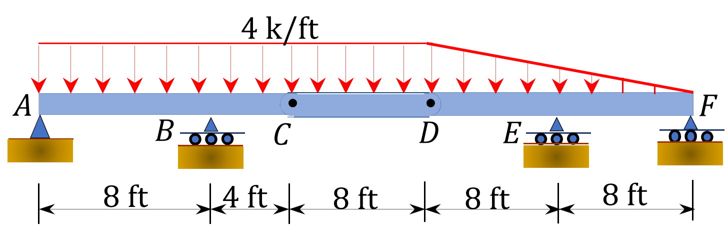


Fig. P3.12. Beam.

3.3. Determine the support reactions for the frames shown in Figure P3.13 through Figure P3.20.

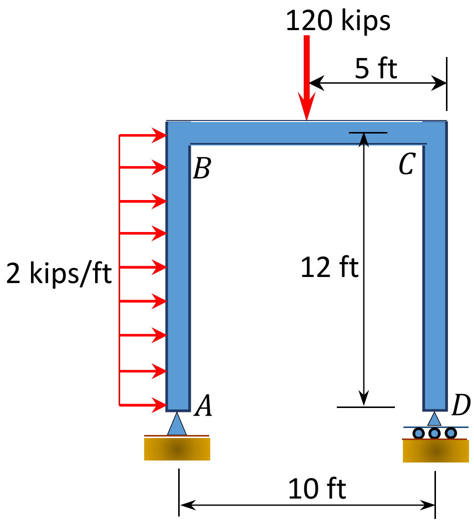


Fig. P3.13. Frame.

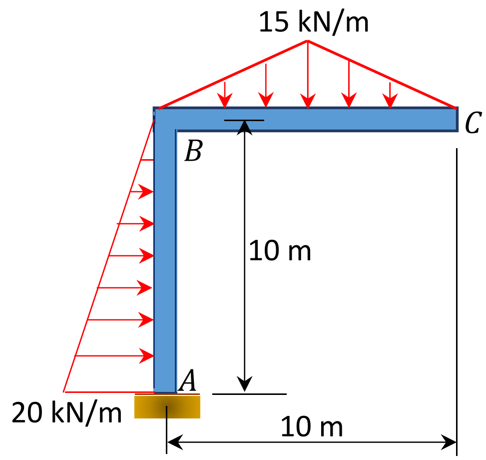


Fig. P3.14. Frame.

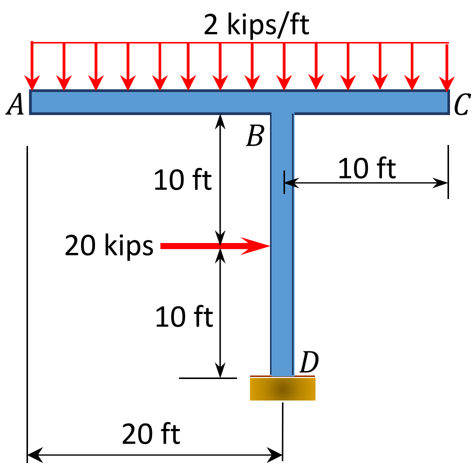


Fig. P3.15. Frame.

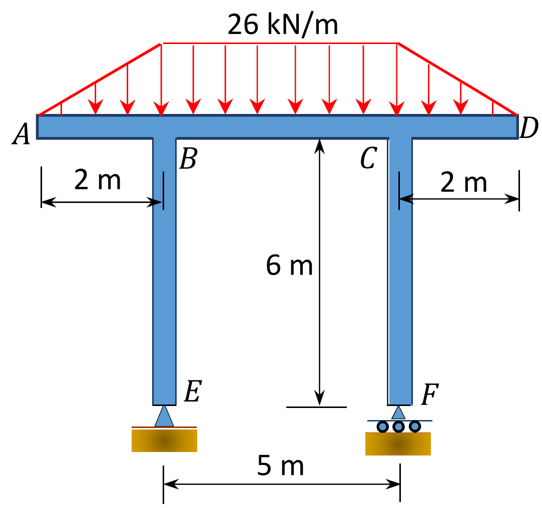


Fig. P3.16. Frame.

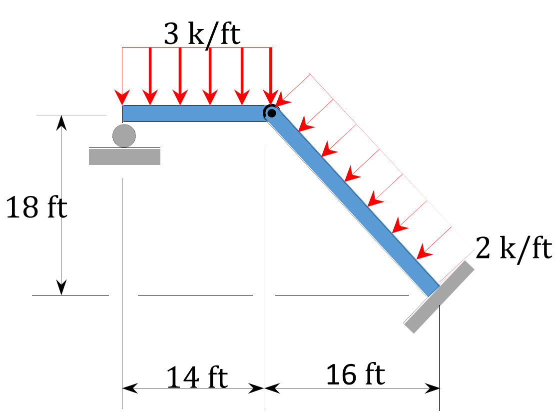


Fig. 3.17. Frame.

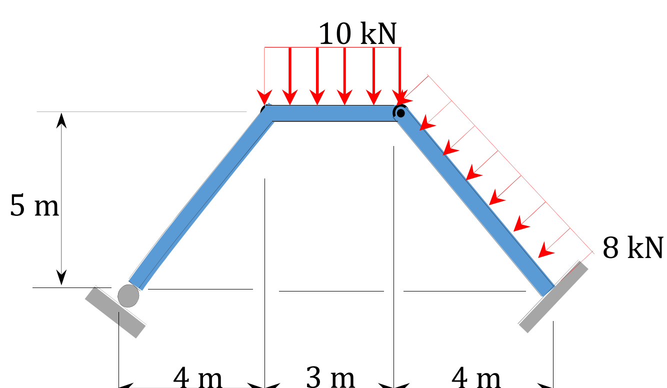


Fig. 3.18. Frame.

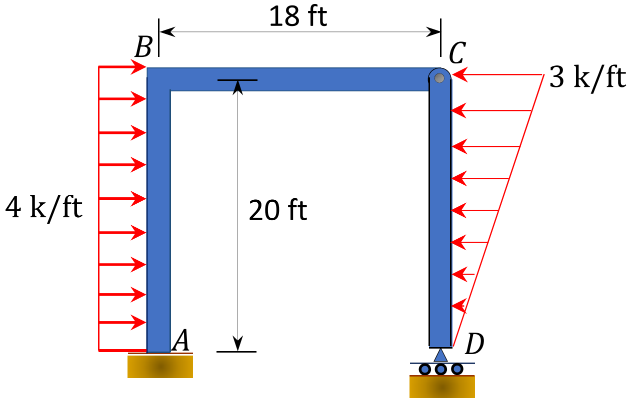


Fig. 3.19. Frame.

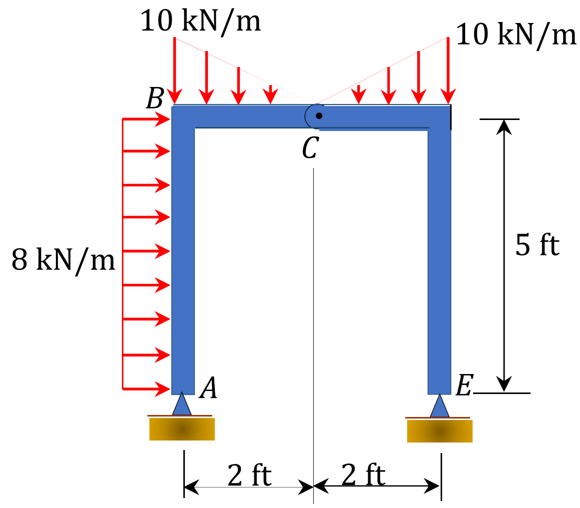


Fig. 3.20. Frame.

3.4 Determine the support reactions for the trusses shown in Figure P3.21 through Figure P3.27.

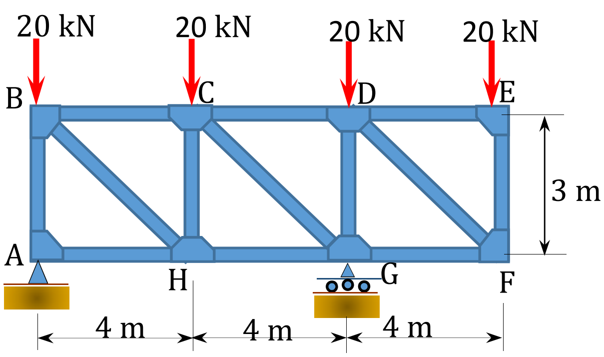


Fig. P3.21. Truss.

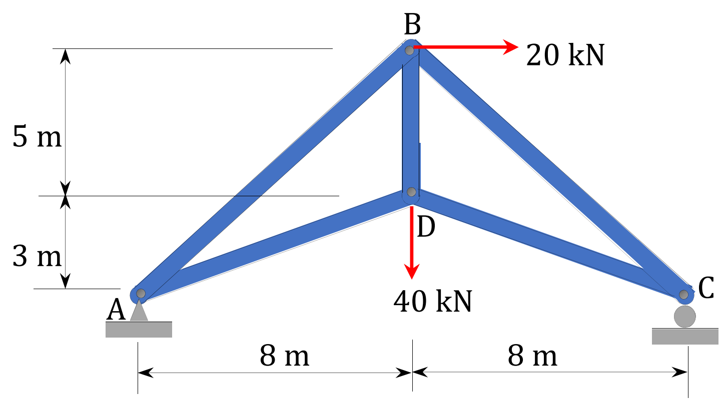


Fig. P3.22. Truss.

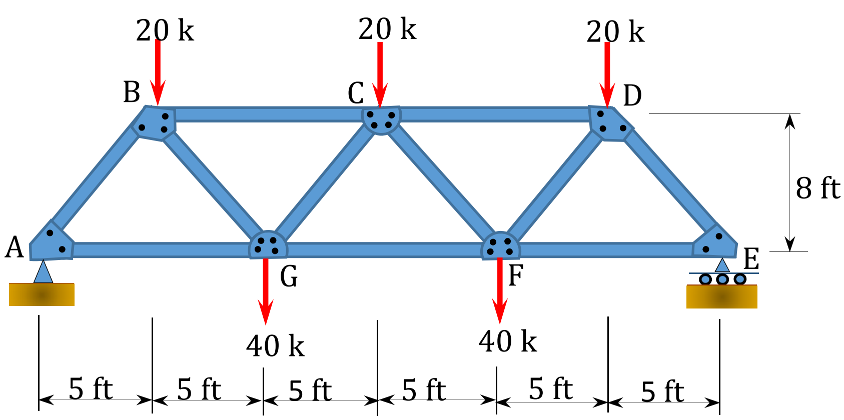


Fig. P3.23. Truss.

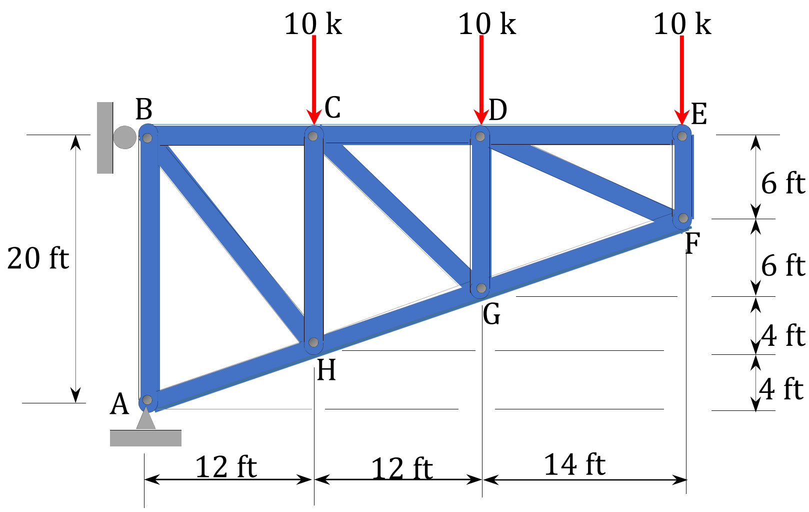


Fig. P3.24. Truss.

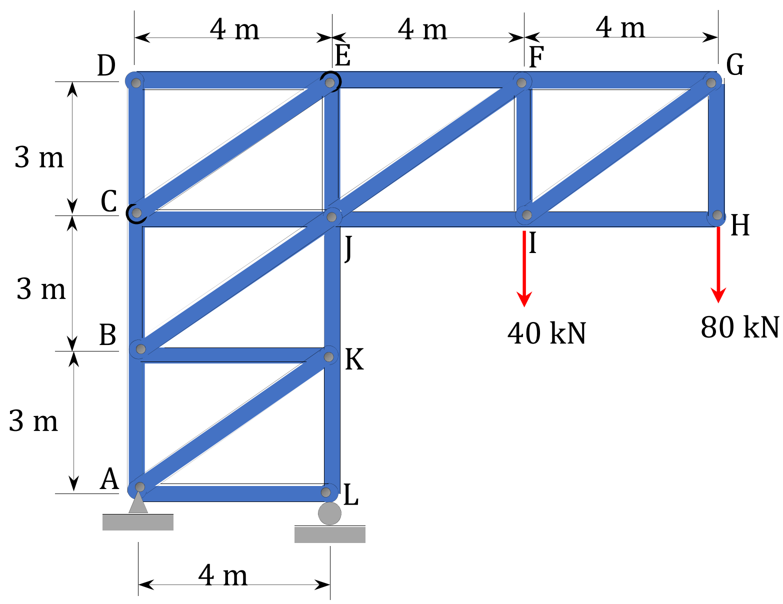


Fig. P3.25. Truss.

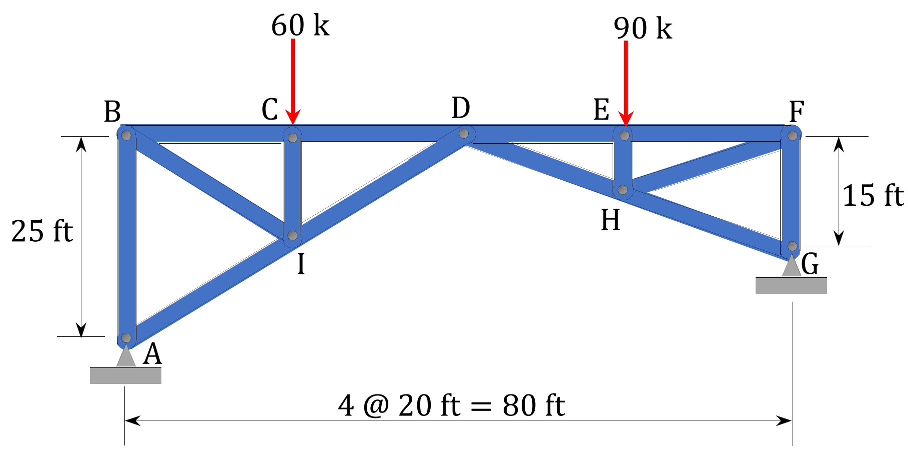


Fig. P3.26. Truss.

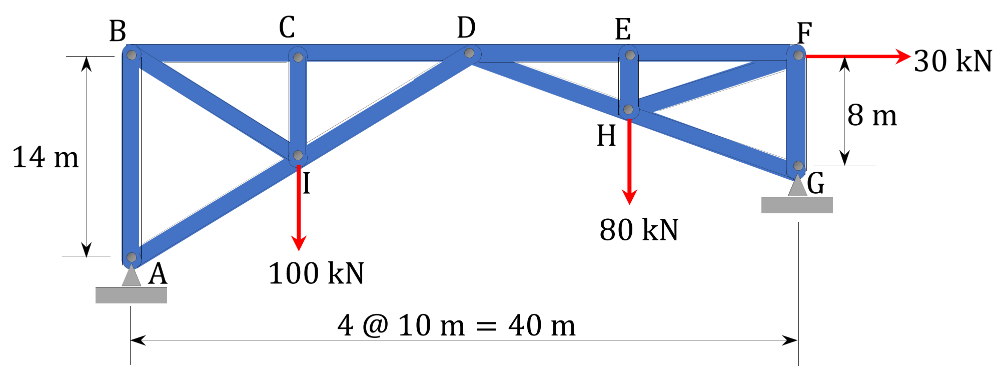


Fig. P3.27. Truss.

Chapter 4

Internal Forces in Beams and Frames

4.1 Introduction

When a beam or frame is subjected to transverse loadings, the three possible internal forces that are developed are the normal or axial force, the shearing force, and the bending moment, as shown in section k of the cantilever of Figure 4.1. To predict the behavior of structures, the magnitudes of these forces must be known. In this chapter, the student will learn how to determine the magnitude of the shearing force and bending moment at any section of a beam or frame and how to present the computed values in a graphical form, which is referred to as the "shearing force" and the "bending moment diagrams." Bending moment and shearing force diagrams aid immeasurably during design, as they show the maximum bending moments and shearing forces needed for sizing structural members.

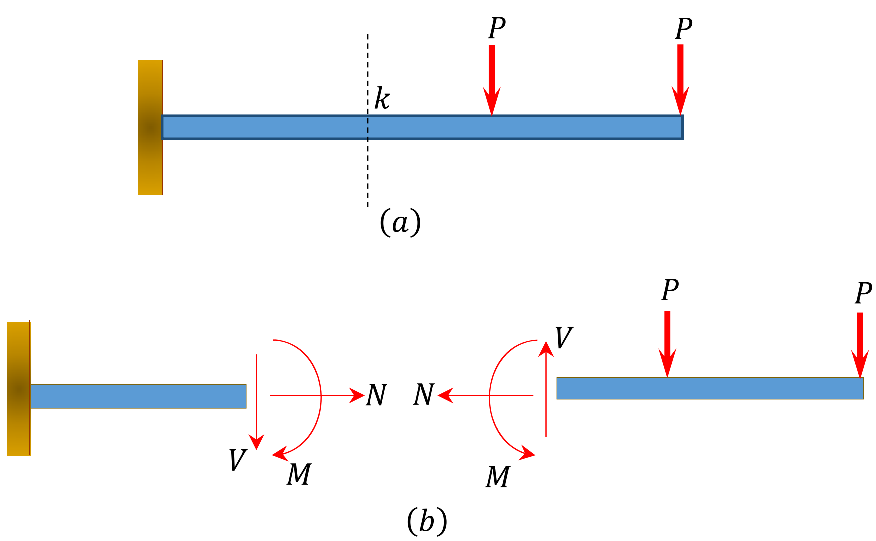


Fig. 4.1. Internal forces in a beam.

4.2 Basic Definitions

4.2.1 Normal Force

The normal force at any section of a structure is defined as the algebraic sum of the axial forces acting on either side of the section.

4.2.2 Shearing Force

The shearing force (SF) is defined as the algebraic sum of all the transverse forces acting on either side of the section of a beam or a frame. The phrase "on either side" is important, as it implies that at any particular instance the shearing force can be obtained by summing up the transverse forces on the left side of the section or on the right side of the section.

4.2.3 Bending Moment

The bending moment (BM) is defined as the algebraic sum of all the forces' moments acting on either side of the section of a beam or a frame.

4.2.4 Shearing Force Diagram

This is a graphical representation of the variation of the shearing force on a portion or the entire length of a beam or frame. As a convention, the shearing force diagram can be drawn above or below the x-centroidal axis of the structure, but it must be indicated if it is a positive or negative shear force.

4.2.5 Bending Moment Diagram

This is a graphical representation of the variation of the bending moment on a segment or the entire length of a beam or frame. As a convention, the positive bending moments are drawn above the x-centroidal axis of the structure, while the negative bending moments are drawn below the axis.

4.3 Sign Convention

4.3.1 Axial Force

An axial force is regarded as positive if it tends to tier the member at the section under consideration. Such a force is regarded as tensile, while the member is said to be subjected to axial tension. On the other hand, an axial force is considered negative if it tends to crush the member at the section being considered. Such force is regarded as compressive, while the member is said to be in axial compression (see Figure 4.2a and Figure 4.2b).

4.3.2 Shear Force

A shear force that tends to move the left of the section upward or the right side of the section downward will be regarded as positive. Similarly, a shear force that has the tendency to move the left side of the section downward or the right side upward will be considered a negative shear force (see Figure 4.2c and Figure 4.2d).

4.3.3 Bending Moment

A bending moment is considered positive if it tends to cause concavity upward (sagging). If the bending moment tends to cause concavity downward (hogging), it will be considered a negative bending moment (see Figure 4.2e and Figure 4.2f).

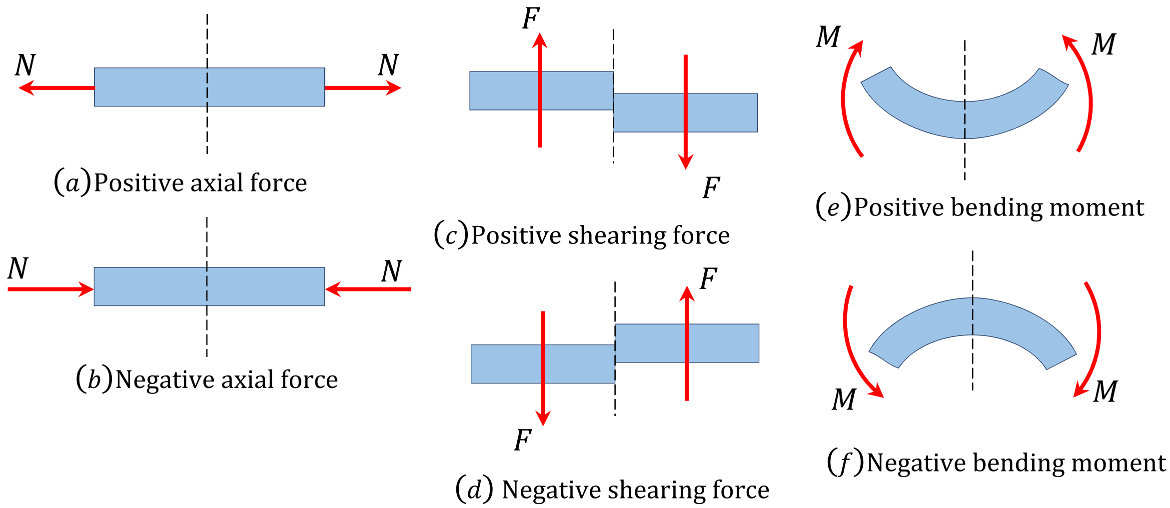


Fig. 4.2. Sign conventions for axial force, shearing force, and bending moment.

4.4 Relation Among Distributed Load, Shearing Force, and Bending Moment

For the derivation of the relations among w, V, and M, consider a simply supported beam subjected to a uniformly distributed load throughout its length, as shown in Figure 4.3. Let the shear force and bending moment at a section located at a distance of x from the left support be V and M, respectively, and at a section $x + dx$ be $V + dV$ and $M + dM$, respectively. The total load acting through the center of the infinitesimal length is wdx.

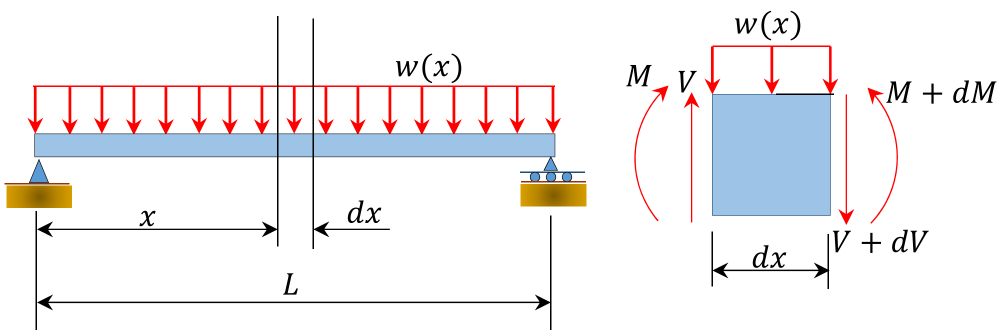


Fig. 4.3. Simply supported beam.

To compute the bending moment at section $x + dx$, use the following:

$$M_{x+dx} = M + Vdx - wdx.dx/2$$

$$= M + Vdx \text{ (neglecting the small second order term } wdx^2/2)$$

$$M + dM = M + Vdx$$

or

$$\frac{dM}{dx} = V(x) \tag{4.1}$$

Equation 4.1 implies that the first derivative of the bending moment with respect to the distance is equal to the shearing force. The equation also suggests that the slope of the moment diagram at a particular point is equal to the shear force at that same point. Equation 4.1 suggests the following expression:

$$\Delta M = \int V(x)dx \tag{4.2}$$

Equation 4.2 states that the change in moment equals the area under the shear diagram. Similarly, the shearing force at section $x + dx$ is as follows:

$$V_{x+dx} = V - wdx$$

$$V + dV = V - wdx$$

or

$$\frac{dV}{dx} = -w(x) \tag{4.3}$$

Equation 4.3 implies that the first derivative of the shearing force with respect to the distance is equal to the intensity of the distributed load. Equation 4.3 suggests the following expression:

$$\Delta V = \int w(x)dx \tag{4.4}$$

Equation 4.4 states that the change in the shear force is equal to the area under the load diagram. Equation 4.1 and 4.3 suggest the following:

$$\frac{d^2M}{dx^2} = -w(x) \tag{4.5}$$

Equation 4.5 implies that the second derivative of the bending moment with respect to the distance is equal to the intensity of the distributed load.

Procedure for Computation of Internal Forces

- Draw the free-body diagram of the structure.
- Check the stability and determinacy of the structure. If the structure is stable and determinate, proceed to the next step of the analysis.
- Determine the unknown reactions by applying the conditions of equilibrium.
- Pass an imaginary section perpendicular to the neutral axis of the structure at the point where the internal forces are to be determined. The passed section divides the structure into two parts. Consider either part of the structure for the computation of the desired internal forces.
- For axial force computation, determine the summation of the axial forces on the part being considered for analysis.
- For shearing force and bending moment computation, first write the functional expression for these internal forces for the segment where the section lies, with respect to the distance x from the origin.
- Compute the principal values of the shearing force and the bending moment at the segment where the section lies.
- Draw the axial force, shearing force, and bending moment diagram for the structure, noting the sign conventions discussed in section 4.3.
- For cantilevered structures, step three could be omitted by considering the free-end of the structure as the initial starting point of the analysis.

Example 4.1

Draw the shearing force and bending moment diagrams for the cantilever beam supporting a concentrated load at the free end, as shown in Figure 4.4a.

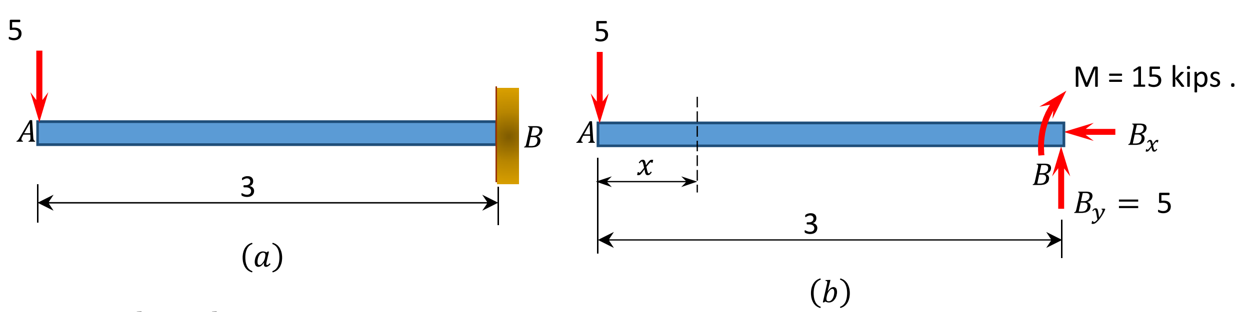
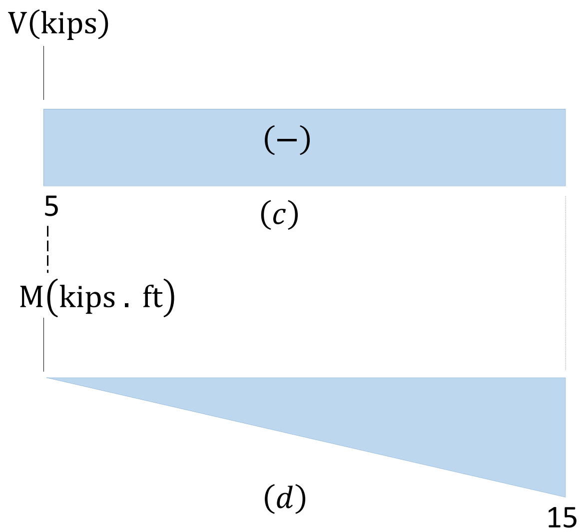


Fig. 4.4. Cantilever beam.



Solution

Support reactions. First, compute the reactions at the support. Since the support at B is fixed, there will be three reactions at that support, namely B_y, B_x, and M_B, as shown in the free-body diagram in Figure 4.4b. Applying the conditions of equilibrium suggests the following:

$$\sum M_B = 0: \quad (5\text{ k})(3\text{ ft}) - M = 0$$

$$M = 15\text{ k.ft}$$

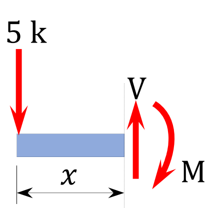
$$\sum F_y = 0: \quad -5\text{k} + B_y = 0$$

$$B_y = 5\text{ k}$$

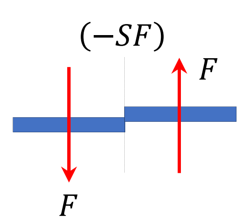
$$\sum F_x = 0: \quad B_x = 0$$

Shearing force (SF).

Shearing force function. Let x be the distance of an arbitrary section from the free end of the cantilever beam (Figure 4.4b). The shearing force at that section due to the transverse forces acting on the segment of the beam to the left of the section (see Figure 4.4e) is $V = -5$ k.



(e)



The negative sign is indicative of a negative shearing force. This is due to the fact that the sign convention for a shearing force states that a downward transverse force on the left of the section under consideration will cause a negative shearing force on that section.

Shearing force diagram. Note that because the shearing force is a constant, it must be of the same magnitude at any point along the beam. As a convention, the shearing force diagram is plotted above or below a line corresponding to the neutral axis of the beam, but a plus sign must be indicated if it is a positive shearing force, and a minus sign should be indicated if it is a negative shearing force, as shown in Figure 4.4c.

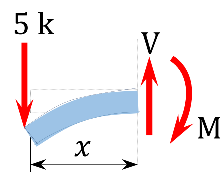
Bending moment (BM).

Bending moment function. By definition, the bending moment at a section is the summation of the moments of all the forces acting on either side of the section. Thus, the expression for the bending moment of the 5 k force on the section at a distance x from the free end of the cantilever beam is as follows:

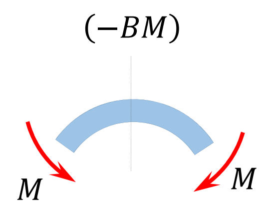
$M = -5x$

When $x = 0, M = -(5\text{ k})(0) = 0$

When $x = 3\text{ ft}$, $M = -(5\text{ k})(3\text{ ft}) = -15\text{ k.ft}$



(f)



The obtained expression is valid for the entire beam (the region $0 < x < 3$ ft). The negative sign indicates a negative moment, which was established from the sign convention for the moment. As seen in Figure 4.4f, the moment due to the 5 k force tends to cause the segment of the beam on the left side of the section to exhibit an upward concavity, and that corresponds to a negative bending moment, according to the sign convention for bending moment.

Bending moment diagram. Since the function for the bending moment is linear, the bending moment diagram is a straight line. Thus, it is enough to use the two principal values of bending moments determined at $x = 0$ ft and at $x = 3$ ft to plot the bending moment diagram. As a convention, negative bending moment diagrams are plotted below the neutral axis of the beam, while positive bending moment diagrams are plotted above the axis of the beam, as shown is Figure 4.4d.

Example 4.2

Draw the shearing force and bending moment diagrams for the cantilever beam subjected to a uniformly distributed load in its entire length, as shown in Figure 4.5a.

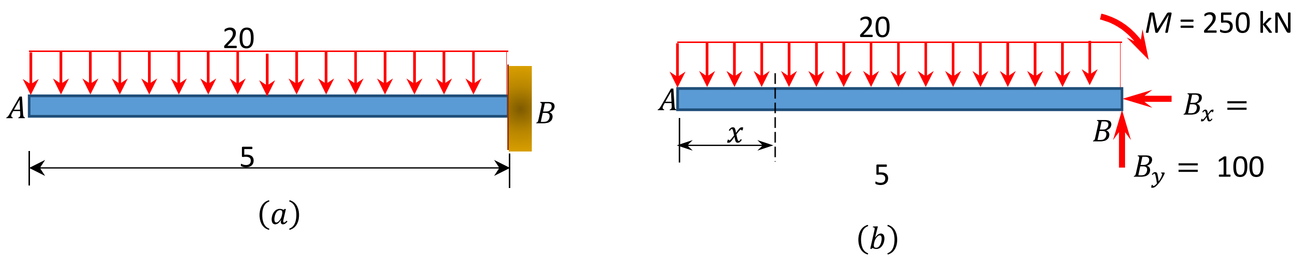
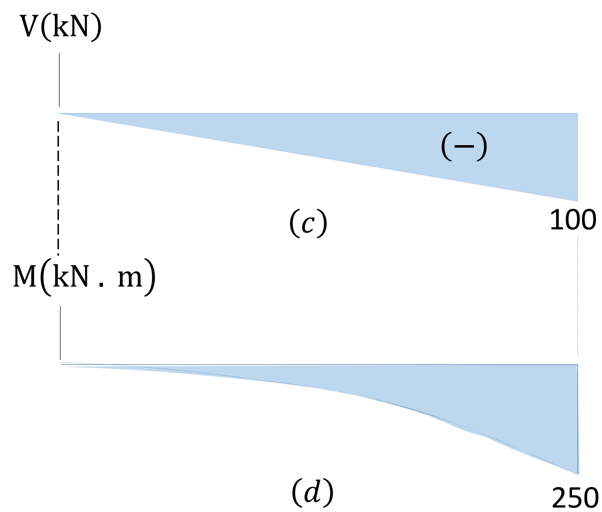


Fig. 4.5. Cantilever beam.



Solution

Support reactions. First, compute the reactions at the support. Since the support at B is fixed, there will possibly be three reactions at that support, namely B_y, B_x, and M_B, as shown in the free-body diagram in Figure 4.4b. Applying the conditions of equilibrium suggests the following:

$$\sum M_B = 0: \quad (20\ \text{kN/m})(5\ \text{m})(2.5\ \text{m}) - M = 0$$

$$M = 250\ \text{kN.m}$$

$$\sum F_y = 0: \quad -(20\ \text{kN/m})(5) + B_y = 0$$

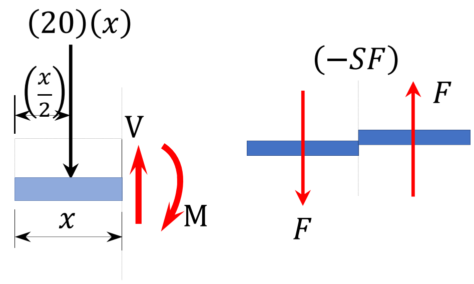
$$B_y = 100\ \text{kN}$$

$$\sum F_x = 0: \quad B_x = 0$$

Shearing force (SF).

Shearing force function. Let x be the distance of an arbitrary section from the free end of the cantilever beam, as shown in Figure 4.5b. The shearing force of all the forces acting on the segment of the beam to the left of the section, as shown in Figure 4.5e, is determined as follows:

$0 < x < 5$
$V = -20x$
When $x = 0$, $V = 0$
When $x = 2.5$ m, $V = -50$ kN
When $x = 5$ m, $V = -100$ kN



(e)

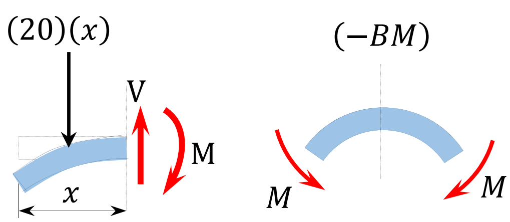
The obtained expression is valid for the entire beam. The negative sign indicates a negative shearing force, which was established from the sign convention for a shearing force. The expression also shows that the shearing force varies linearly with the length of the beam.

Shearing force diagram. Note that because the expression for the shearing force is linear, its diagram will consist of straight lines. The shearing force at $x = 0$ m and $x = 5$ m were determined and used for plotting the shearing force diagram, as shown in Figure 4.5c. As shown in the diagram, the shearing force varies from zero at the free end of the beam to 100 kN at the fixed end. The computed vertical reaction of B_y at the support can be regarded as a check for the accuracy of the analysis and diagram.

Bending moment (BM).

Bending moment expression. The expression for the bending moment at a section of a distance x from the free end of the cantilever beam is as follows:

$0 < x < 5$ m
$M = -\frac{20x^2}{2}$
When $x = 0$, $M = 0$
When $x = 2.5$ m, $M = -62.5$ kN.m
When $x = 5$ m, $M = -250$ kN.m



(f)

The negative sign indicates a negative moment, which was established from the sign convention for moment. As seen in Figure 4.5f, the moment due to the distributed load tends to cause the segment of the beam on the left side of the section to exhibit an upward concavity, and that corresponds to a negative bending moment, according to the sign convention for bending moment.

Bending moment diagram. Since the function for the bending moment is parabolic, the bending moment diagram is a curve. In addition to the two principal values of bending moment at $x = 0$ m and at $x = 5$ m, the moments at other intermediate points should be determined to correctly draw the bending moment diagram. The bending moment diagram of the beam is shown in Figure 4.5d.

Example 4.3

Draw the shearing force and bending moment diagrams for the cantilever beam subjected to the loads shown in Figure 4.6a.

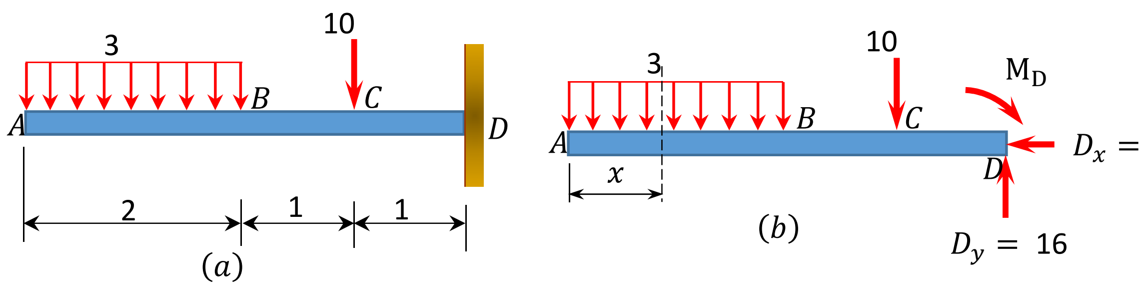
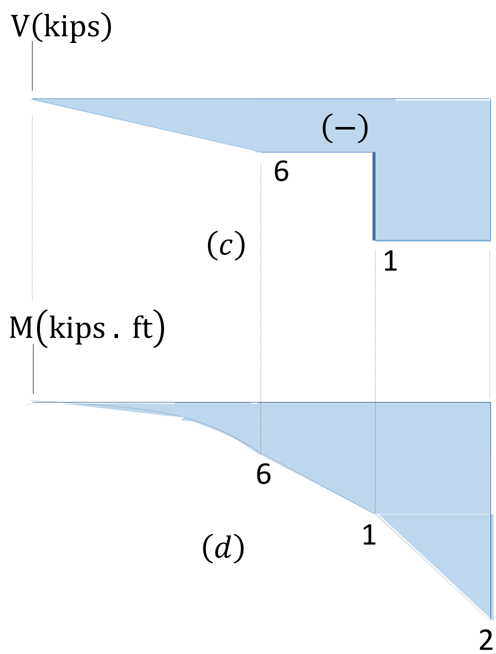


Fig. 4.6. Cantilever beam.



Solution

Support reactions. The free-body diagram of the beam is shown in Figure 4.6b. First, compute the reactions at the support B. Applying the conditions of equilibrium suggests the following:

$$\sum M_B = 0: (3\text{ k/ft})(2\text{ ft})(3\text{ m}) + (10\text{ k})(1) - M = 0$$

$$M = 28\text{ k.ft}$$

$$\sum F_y = 0: -\left(3\frac{\text{k}}{\text{ft}}\right)(2\text{ ft}) - 10\text{ k} + D_y = 0$$

$$D_y = 16\text{ k}$$

$$\sum F_x = 0: D_x = 0$$

Shearing force and bending moment functions. Due to the discontinuity of the distributed load at point *B* and the presence of the concentrated load at point *C*, three regions describe the shear and moment functions for the cantilever beam. The functions and the values for the shear force (*V*) and the bending moment (*M*) at sections in the three regions at a distance x from the free-end of the beam are as follows:

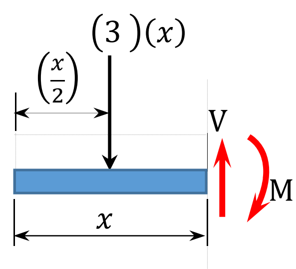
Segment *AB* $0 < x < 2$ ft

$V = -3x$

When $x = 0, V = 0$

When $x = 1, V = -3$ kip

When $x = 2$ ft, $V = -6$ kip



(e)

$M = -\frac{3x^2}{2}$

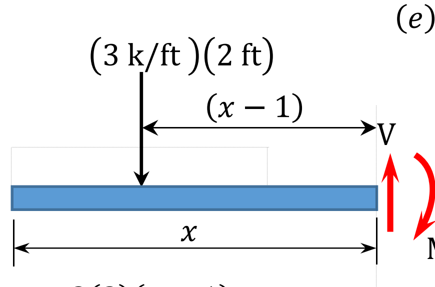
When $x = 0, M = 0$

When $x = 1$ ft, $M = -1.5$ kip.ft

When $x = 2$ ft, $M = -6$ kip.ft

Segment *BC* 2 ft $< x <$ 3 ft

$V = -3(2) = -6$ kip



$M = -3(2)(x-1)$ (f)

When $x = 2$ ft, $M = -6$ kip.ft

When $x = 3$ ft, $M = -12$ kip.ft

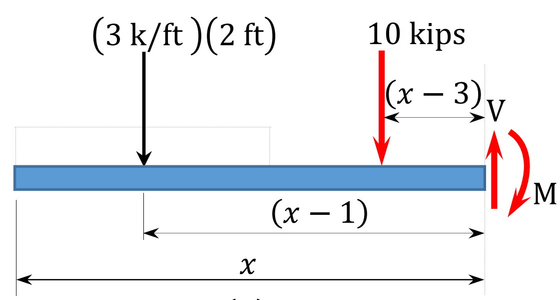
Segment *CD* 3 ft $< x <$ 4 ft

$V = -(3)(2) - 10 = -16$ kips

$M = -(3)(2)(x-1) - 10(x-3)$

When $x = 3$ ft, $M = -12$ kip.ft

When $x = 4$ ft, $M = -28$ kip.ft



(g)

The computed shearing force can be checked in part with the support reactions shown on the free-body diagram in Figure 4.6b.

Shearing force and bending moment diagrams. The computed values of the shearing force and bending moment are plotted in Figure 4.6c and Figure 4.6d. It is important to remember that there will always be a sudden change in the shearing force diagram where there is a concentrated load in the beam. The numerical value of the change should be equal to the value of the concentrated load. For instance, at point *C* where the concentrated load of 10 kips is located in the beam, the change in shearing force in the shear force diagram is 16 k - 6k = 10 kips. The bending moment diagram is a curve in portion *AB* and is straight lines in segments *BC* and *CD*.

Example 4.4

Draw the shearing force and bending moment diagrams for the beam with an overhang subjected to the loads shown in Figure 4.7a.

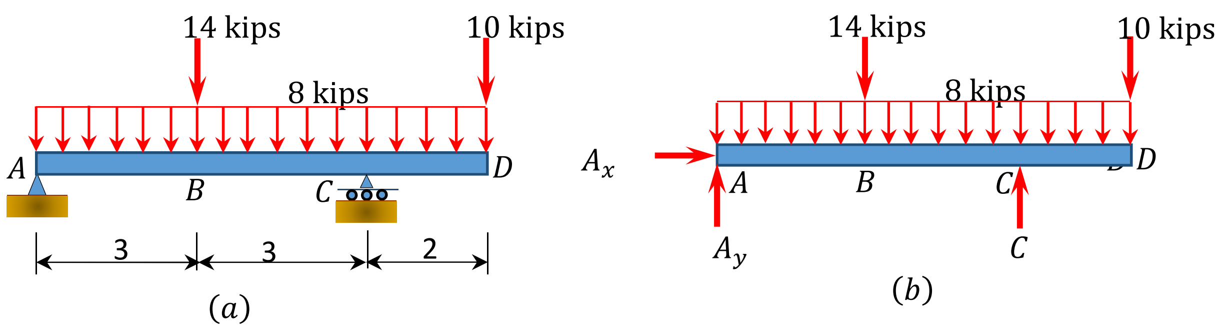
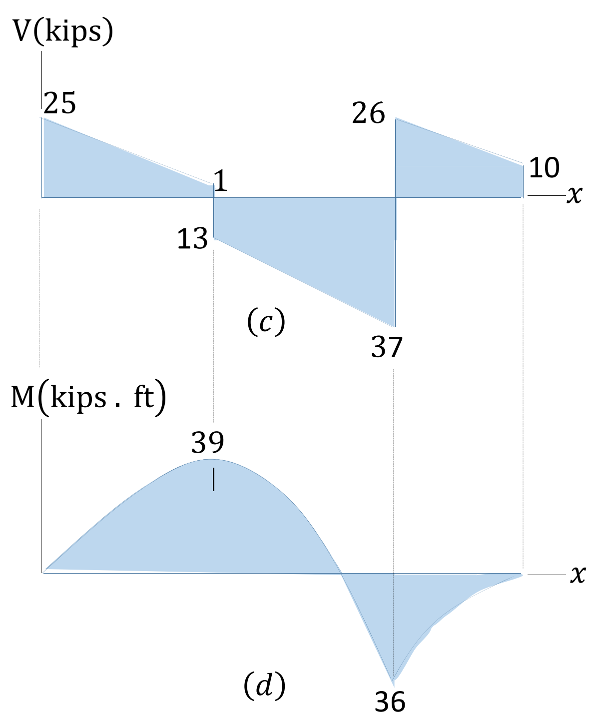


Fig. 4.7. Beam with an overhang.



Solution

Support reactions. The reactions at the supports are shown in the free-body diagram of the beam in Figure 4.7b. They are computed by applying the conditions of equilibrium, as follows:

$$+\curvearrowleft \sum M_A = 0$$
$$-(14)(3) - (10)(8) - (8)(8)(4) + B_y(6) = 0$$
$B_y = 63$ kips $\qquad B_y = 63 \uparrow$

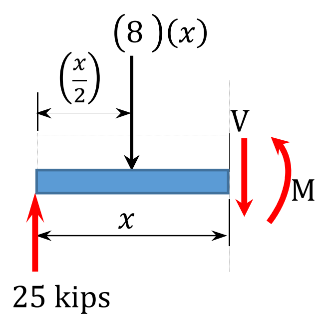
$+\rightarrow \sum F_x = 0 \quad A_x = 0 \qquad A_x = 0$

$+\uparrow \sum F_y = 0$
$$63 + A_y - 14 - 10 - (8)(8) = 0$$
$A_y = 25$ kips $\qquad A_y = 25$ kips $\uparrow$

Shear and bending moment functions. Due to the concentrated load at point *B* and the overhanging portion *CD*, three regions are considered to describe the shearing force and bending moment functions for the overhanging beam. The expression for these functions at sections within each region and the principal values at the end points of each region are as follows:

$0 < x < 3$
$V = 25 - 8x$
When $x = 0, V = 25$ kips
When $x = 3, V = 1$ kip

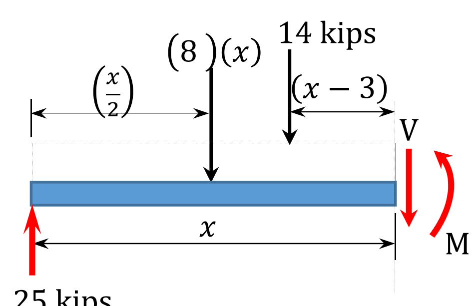
$M = 25x - \frac{8x^2}{2}$
When $x = 0, M = 0$
When $x = 3, M = 39$ kip.ft



(e)

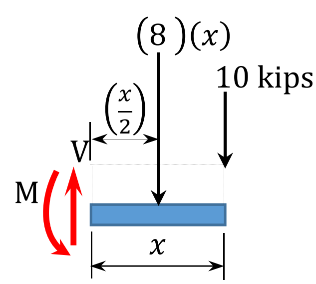
$3 < x < 6$
$V = 25 - 14 - 8x$
When $x = 3, V = -13$ kips
When $x = 6, V = -37$ kips
$M = 25x - 14(x-3) - \frac{8x^2}{2}$

When $x = 3, M = 39$ k.ft
When $x = 6, M = -36$ kip.ft



(f)

$0 < x < 2$
$V = 10 + 8x$
When $x = 0, V = 10$ kips
When $x = 2, V = 26$ kips



(g)

$M = 10x - \frac{8x^2}{2}$

When $x = 0, M = 0$

When $x = 2, M = -36$ kip. ft

Shearing force and bending moment diagram. The determined shearing force and moment diagram at the end points of each region are plotted in Figure 4.7c and Figure 4.7d. For accurate plotting of the bending moment curve, it is sometimes necessary to determine some values of the bending moment at intermediate points by inserting some distances within the region into the obtained function for that region. Notice that at the location of concentrated loads and at the supports, the numerical values of the change in the shearing force are equal to the concentrated load or reaction.

Example 4.5

Draw the shearing force and bending moment diagrams for the beam with an overhang subjected to the loads shown in Figure 4.8a. Determine the position and the magnitude of the maximum bending moment.

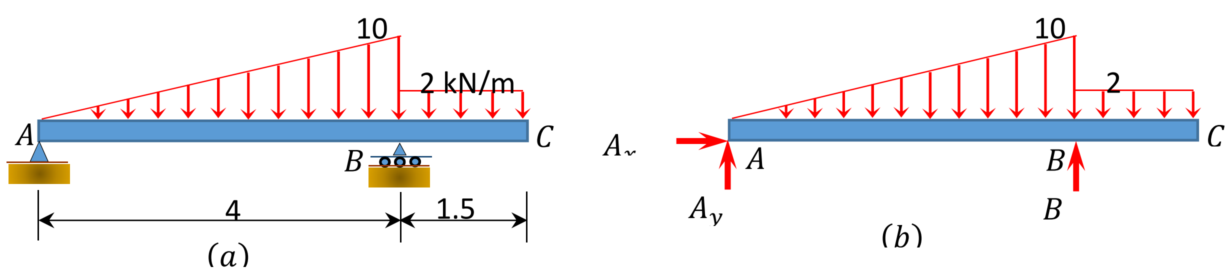
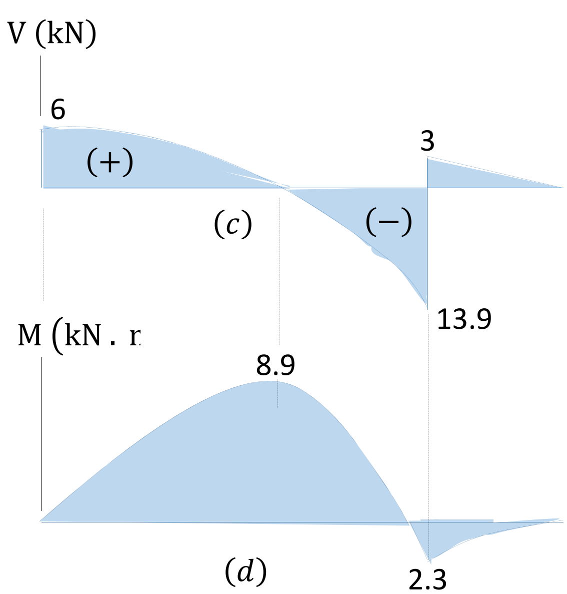


Fig. 4.8. Beam with an overhang.



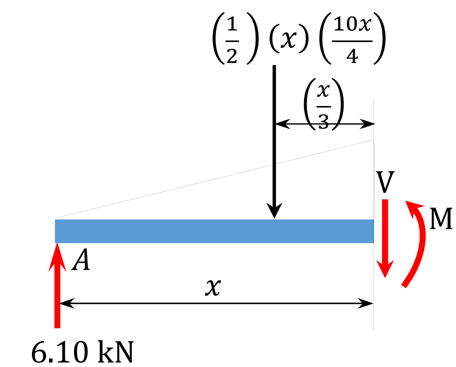
Solution

Support reactions. The reactions at the supports of the beam are shown in the free-body diagram in Figure 4.8b. The reactions are computed by applying the following equations of equilibrium:

$$+\circlearrowleft \sum M_A = 0$$
$$-\left(\frac{1}{2}\right)(4)(10)\left(\frac{2}{3}\times 4\right) - (2)(1.5)(4.75) + (4)B_y = 0$$
$$B_y = 16.90 \text{ kN} \uparrow$$
$$+\uparrow \sum F_y = 0$$
$$A_y + 16.90 - \left(\frac{1}{2}\right)(4)(10) - (2)(1.5) = 0$$
$$A_y = 6.10 \text{ kN} \uparrow$$

$$+\rightarrow \sum F_x = 0$$
$$A_x = 0$$

Shear and bending moment functions. Due to the discontinuity in the shades of distributed loads at the support B, two regions of x are considered for the description and moment functions, as shown below:

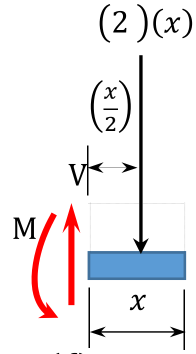


(e)

$$0 < x < 4$$
$$V = 6.10 - \left(\frac{1}{2}\right)(x)\left(\frac{10x}{4}\right)$$
When $x = 0, V = 6.10$ kN
When $x = 2, V = 1.1$ kN

When $x = 4, V = -13.9$ kN

$$M = 6.10x - \left(\frac{1}{2}\right)(x)\left(\frac{10x}{4}\right)\left(\frac{1}{3}x\right)$$
When $x = 0, M = 0$
When $x = 2, M = 8.87$ kN. m
When $x = 4, M = -2.3$ kN. m



(f)

$$0 < x < 1.5$$
$$V = 2x$$
When $x = 0, V = 0$
When $x = 1.5, V = 3$ kN

$$M = -(2)(x)\left(\frac{x}{2}\right)$$
When $x = 0, M = 0$
When $x = 1.5$ m, $M = -2.3$ kN. m

Shearing force and bending moment diagrams. The computed values of the shearing force and bending moment are plotted in Figure 4.8c and Figure 4.8d. Observe that the values of the shear force at the supports are equal to the values of the support reactions. Also, notice in the diagram that the shear in the region AB is a curve and the shear in the region BC is a straight, which all correspond to the parabolic and linear functions respectively obtained for the regions. The bending moment diagrams for both regions are curvilinear. The curve for the AB region is deeper than that in the BC region. This is because the obtained function for the AB region is cubical while that for the BC region is parabolic.

Position and magnitude of maximum bending moment. Maximum bending moment occurs where the shearing force equals zero. As shown in the shearing force diagram, the maximum bending moment occurs in the portion AB. Equating the expression for the shear force for that portion as equal to zero suggests the following:

$$V = 6.10 - \frac{10x^2}{8} = 0,$$

$$x = \sqrt{\frac{(6.1)(8)}{10}} = 2.21 \text{ m}$$

The magnitude of the maximum bending moment can be determined by putting $x = 2.21$ m into the expression for the bending moment for the portion AB. Thus,

$$M_{max} = 6.10x - \left(\frac{1}{2}\right)(x)\left(\frac{10x}{4}\right)\left(\frac{1}{3}x\right) = (6.1)(2.21) - \frac{(10)(2.21^3)}{24} = 8.98 \text{ kN.m}$$

Example 4.6

Draw the shearing force and bending moment diagrams for the compound beam subjected to the loads shown in Figure 4.9a.

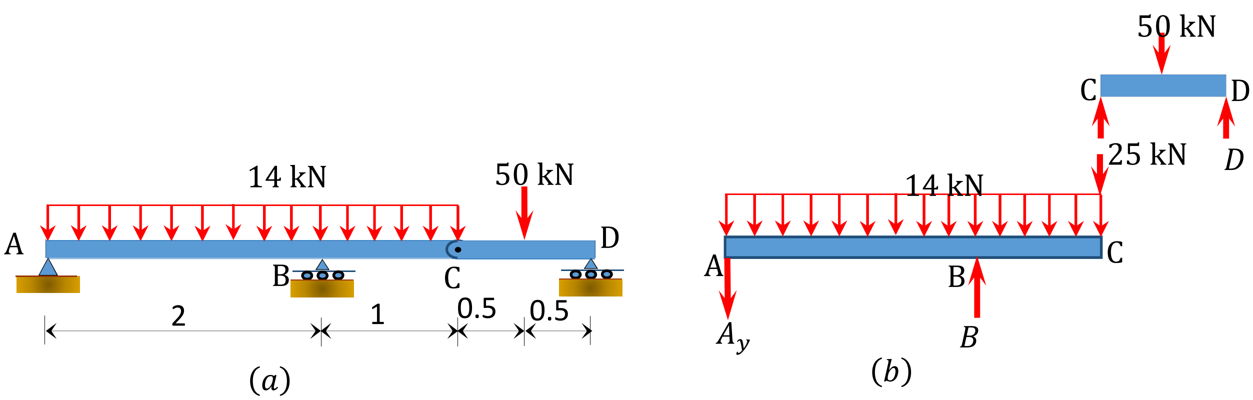
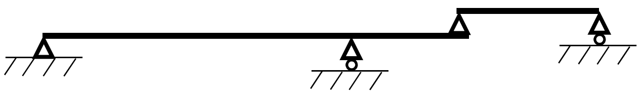
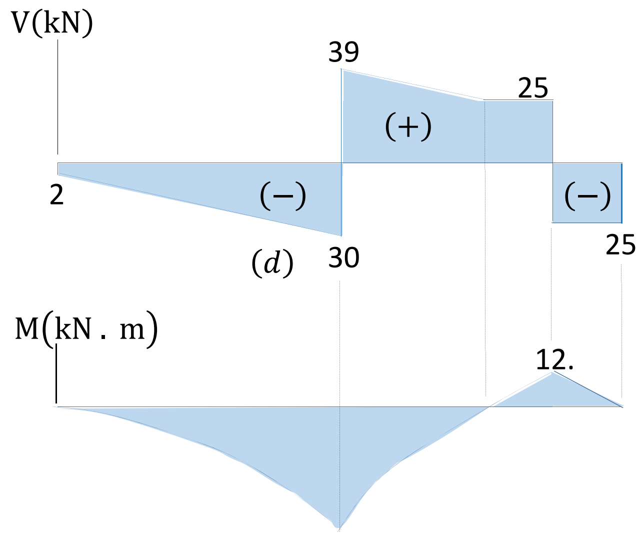


Fig. 4.9. Compound beam.



(*c*)



(*d*)

(*e*)

Solution

Free-body diagram. The free-body diagram of the beam is shown in Figure 4.9b.

Classification of structure. The compound beam has $r = 4, m = 2$, and $f_i = 2$. Since $4 + 2 = 3(2)$, the structure is statically determinate.

Identification of the primary and complimentary structure. The schematic diagram of member interaction for the beam is shown in Figure 4.9c. The part *AC* is the primary structure, while part *CD* is the complimentary structure.

Analysis of complimentary structure.

Support reaction.

$C_y = D_y = 25$ kN, due to symmetry of loading.

Shear force and bending moment.

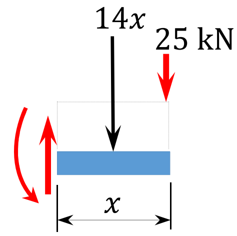
$0 < x < 0.5$

$V = 25$ kN

$M = 25x$

When $x = 0, M = 0$

When $x = 0.5, M = 12.5$ kN. m



(*f*)

Analysis of primary structure.

Support reactions.

$+\curvearrowleft \sum M_A = 0$

$2B_y - (14)(3)(1.5) - (25)(3) = 0$

$B_y = 69 \text{ kN} \uparrow$

$+\uparrow \sum F_y = 0$

$69 + A_y - 25 - (14)(3) = 0$

$A_y = -2 \text{ kN}$

The negative implies the reaction at A acts downward.

$+\rightarrow \sum F_x = 0$

$A_x = 0$

Shear force and bending moment functions.

$0 < x < 1$

$V = 25 + 14x$

When $x = 0, V = 25 \text{ kN}$

When $x = 1, V = 39 \text{ kN}$

$M = -25x - \frac{14x^2}{2}$

When $x = 0, M = 0$

When $x = 1, M = -32 \text{ kN.m}$

$0 < x < 2$

$V = -2 - 14x$

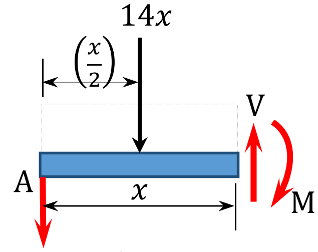
When $x = 0, V = -2 \text{ kN}$

When $x = 2, V = -30 \text{ kN}$

$M = -2x - \frac{14x^2}{2}$

When $x = 0, M = 0$

When $x = 2, M = -32 \text{ kN.m}$



$A_y = 2 \text{ kN}$

(g)

Shearing force and bending moment diagrams. The computed values of the shearing force and bending moment for the primary and complimentary part of the compound beam are plotted in Figure 4.9d and Figure 4.9e.

Example 4.7

Draw the shear force and bending moment diagrams for the frame subjected to the loads shown in Figure 4.10a.

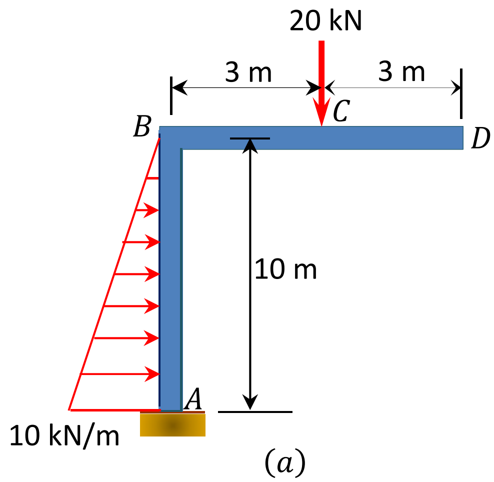
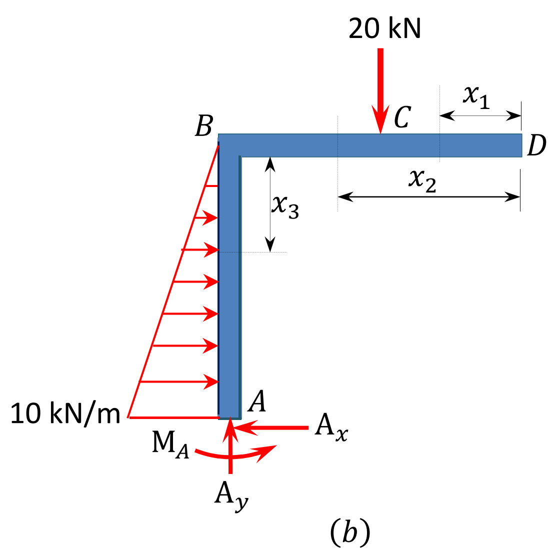
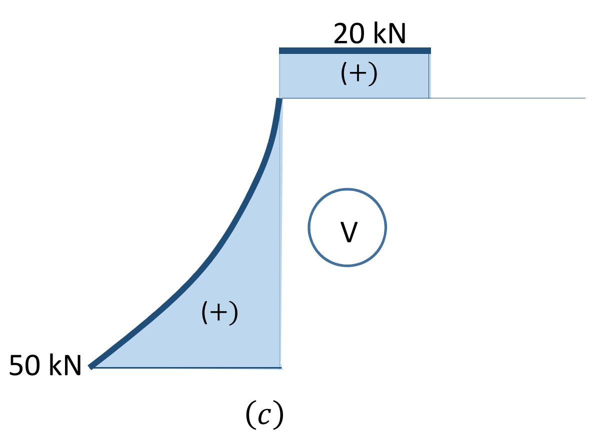
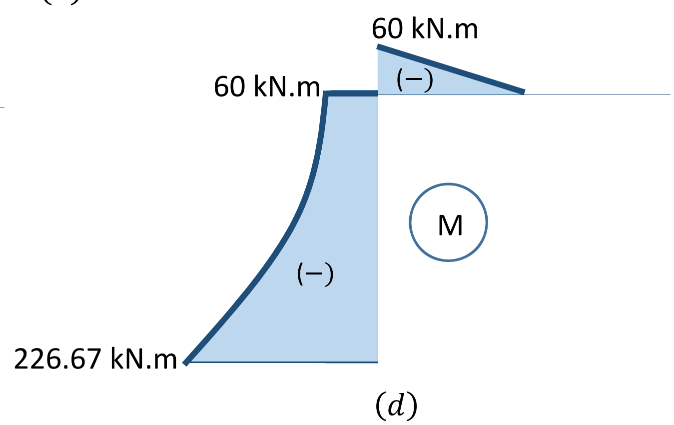


Fig. 4.10. Frame.







Solution

Free-body diagram. The free-body diagram of the beam is shown in Figure 4.10a.

Support reactions. The reactions at the support of the beam can be computed as follows when considering the free-body diagram and using the equations of equilibrium:

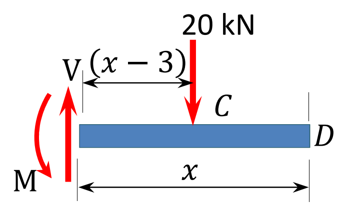
$+\uparrow \sum F_y = 0$
$A_y - 20 = 0$
$A_y = 20 \text{ kN} \uparrow$

$+\rightarrow \sum F_x = 0$
$-A_x + \left(\frac{1}{2} \times 10 \times 10\right) = 0$
$A_x = 50 \text{ kN} \leftarrow$

$+\circlearrowleft \sum M_A = 0$
$M_A - 20(3) - \left(\frac{1}{2} \times 10 \times 10\right)\left(\frac{1}{3} \times 10\right) = 0$
$M_A = 226.67 \text{ kN.m} \circlearrowleft$

Shearing force and bending moment functions of beam *BC*.

$0 < x_1 < 3$
$V = 0$
$M = 0$

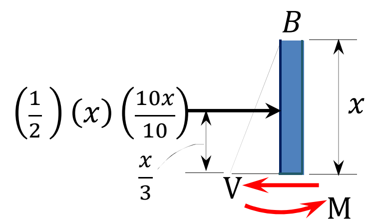


(e)

$3 < x_2 < 6$
$V = 20 \text{ kN}$
$M = -20\,(x - 3)$
When $x = 3$, $M = 0$
When $x = 6$, $M = -60 \text{ kN.m}$
Note that the distance x to the section in the expressions is from the right end of the beam.

Shearing force and bending moment functions of column *AB*.

$0 < x_3 < 10$
$V = \left(\frac{1}{2} \times x \times x\right)$
When $x = 0$, $V = 0$
When $x = 10$, $V = 50 \text{ kN}$



(f)

$M = -20\,(3) - \left(\frac{1}{2} \times x \times x\right)\left(\frac{x}{3}\right)$
When $x = 0$, $M = -60 \text{ kN.m}$
When $x = 10$, $M = -226.67 \text{ kN.m}$

Note that the distance x to the section on the column is from the top of the column and that a similar triangle was used to determine the intensity of the triangular loading at the section in the column, as follows: $\frac{x}{10} = \frac{w}{(10)}$ or $w = \frac{(10x)}{10}$.

Shearing force and bending moment diagrams. The computed values of the shearing force and bending moment for the frame are plotted as shown in Figure 4.10c and Figure 4.10d.

Example 4.8

Draw the shearing force and bending moment diagrams for the frame subjected to the loads shown in Figure 4.11a.

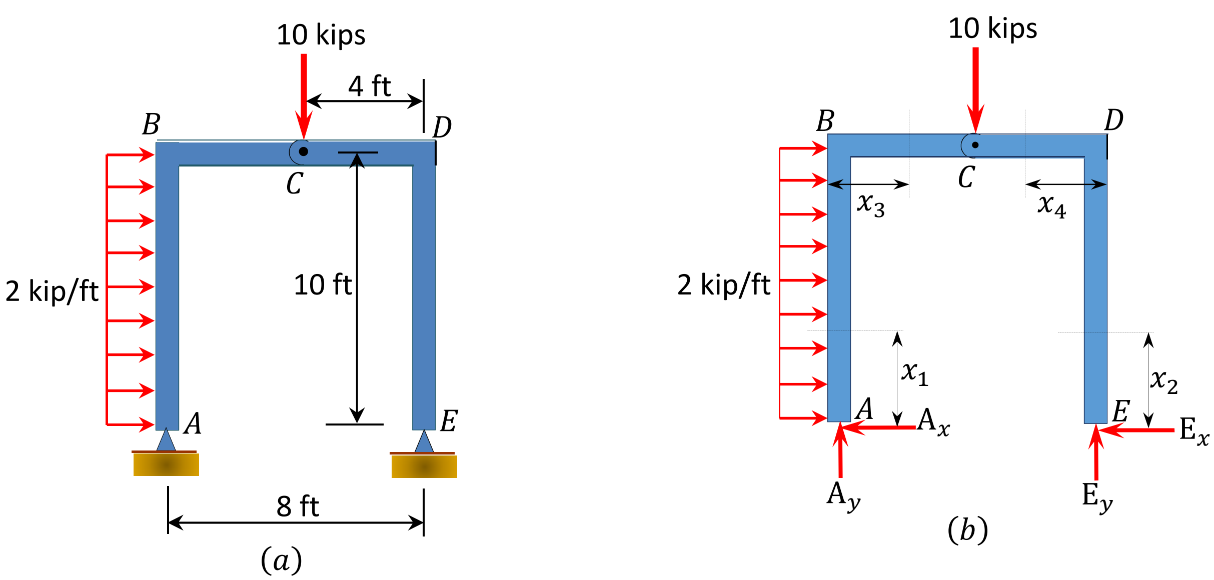
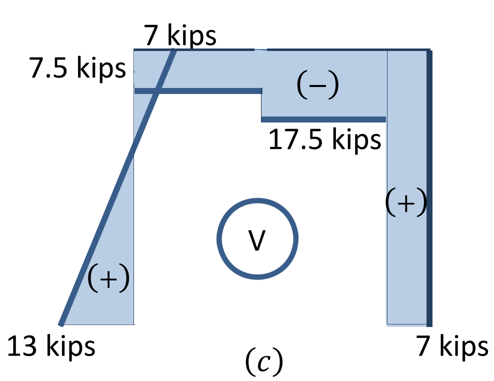
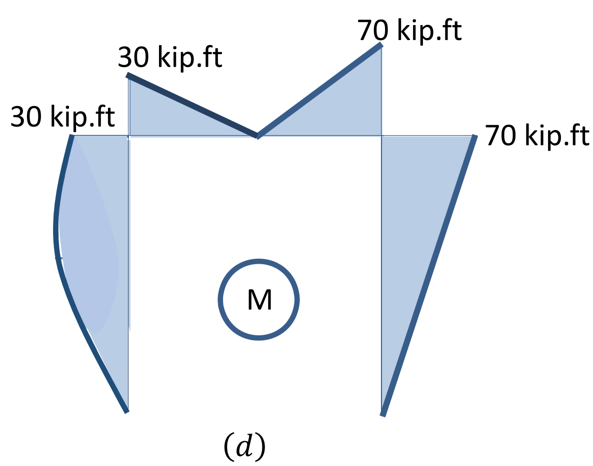


Fig. 4.11. Frame.





Solution

Free-body diagram. The free-body diagram of the beam is shown in Figure 4.11b.

Support reactions. The reactions at the supports of the frame can be computed by considering the free-body diagram of the entire frame and part of the frame. The vertical reactions of the supports at points A and E are computed by considering the equilibrium of the entire frame, as follows:

$+\curvearrowleft \sum M_A = 0$

$-2(10)\left(\frac{10}{2}\right) - 10(4) + E_y(8) = 0$

$E_y = 17.5 \text{ kips}$ $\qquad$ $E_y = 17.5 \text{ kips} \uparrow$

$+\uparrow \sum F_y = 0$

$A_y + 17.5 - 10 = 0$

$A_y = -7.5 \text{ kips}$ $\qquad$ $A_y = 7.5 \text{ kips} \downarrow$

The negative sign indicates that A_y acts downward instead of upward as originally assumed.

Considering the equilibrium of part *CDE* of the frame, the horizontal reaction of the support at E is determined as follows:

$+\curvearrowleft \sum M_C = 0$

$17.5(4) - E_x(10) = 0$

$E_x = 7 \text{ kips} \leftarrow$ $\qquad$ $E_x = 7 \text{ kips} \leftarrow$

Again, considering the equilibrium of the entire frame, the horizontal reaction at A can be computed as follows:

$+\rightarrow \sum F_x = 0$

$-A_x + 2(10) - 7 = 0$

$A_x = 13 \text{ kips} \leftarrow$ $\qquad$ $A_x = 13 \text{ kips} \leftarrow$

Shear and bending moment of the columns of the frame.

Shear force and bending moment in column *AB*.

$0 < x_1 < 10$ ft

$V = 13 - 2x$

When $x = 0$, $V = 13$ kips

When $x = 10$ ft, $V = -7$ kips

$M = 13x - 2\left(\frac{x^2}{2}\right)$

When $x = 0$, $M = 0$

When $x = 10$ ft, $M = 30$ kip.ft

When $x = 5$ ft, $M = 30$ kip. ft

Shear force and bending moment in column *ED*.

$0 < x_2 < 10$ ft
$V = 7$ kips

$M = 7x$
When $x = 0$, $M = 0$
When $x = 10$ ft, $M = 70$ kip. ft

Shear and bending moment of the frame's beam.

Shear force and bending moment in beam *BC*.

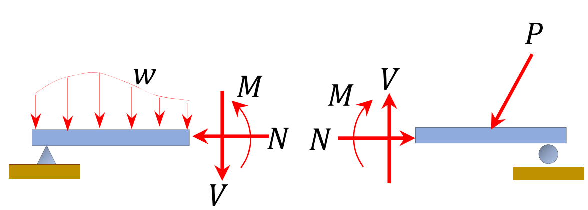
$0 < x_3 < 4$ ft
$V = -7.5$ kips
$M = -7.5x + 13(10) - 2(10)\left(\frac{10}{2}\right)$
When $x = 0$, $M = 30$ kip.ft
When $x = 4$ ft, $M = 0$

Shear force and bending moment in beam *CD*.

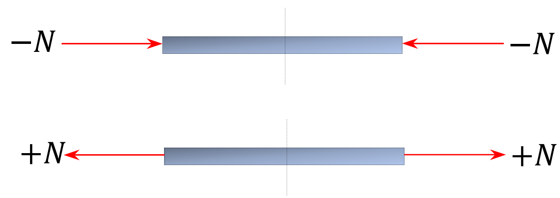
$0 < x_4 < 4$ ft
$V = -17.5$ kips
$M = 17.5x - 7(10)$
When $x = 0$, $M = -70$ kip.ft
When $x = 4$ ft, $M = 0$
The computed values of the shearing force and bending moment for the frame are plotted in Figure 4.11c and Figure 4.11d.

Chapter Summary

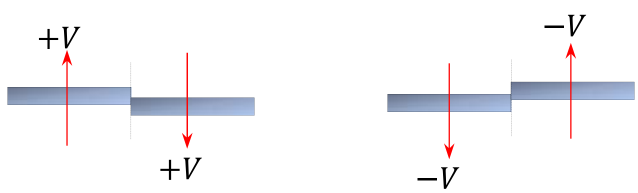
Internal forces in beams and frames: When a beam or frame is subjected to external transverse forces and moments, three internal forces are developed in the member, namely the normal force (*N*), the shear force (*V*), and the bending moment (*M*). These are shown in the following Figure.



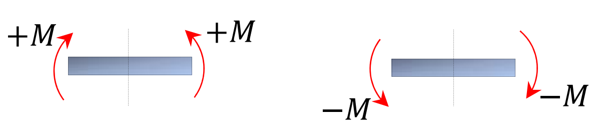
Normal force: The normal force at any section of a beam can be determined by adding up the horizontal, normal forces acting on either side of the section. If the resultant of the normal force tends to move towards the section, it is regarded as compression and is denoted as negative. However, if it tends to move away from the section, it is regarded as tension and is denoted as positive.



Shear force: The shear force at any section of a beam is determined as the summation of all the transverse forces acting on either side of the section. The sign convention adopted for shear forces is below. A diagram showing the variation of the shear force along a beam is called the shear force diagram.



Bending moment: The bending moment at a section of a beam can be determined by summing up the moment of all the forces acting on either side of the section. The sign convention for bending moments is shown below. A graphical representation of the bending moment acting on the beam is referred to as the bending moment diagram.



Relationship among distributed load, shear force, and bending moment: The following relationship exists among distributed loads, shear forces, and bending moments.

$$\frac{dV}{dx} = w$$

$$\Delta V = \int w \, dx$$

$$\frac{dM}{dx} = V$$

$$\Delta M = \int V \, dx$$

$$\frac{d^2M}{dx^2} = W$$

Practice Problems

4.1. Draw the shearing force and the bending moment diagrams for the beams shown in Figure P4.1 through Figure P4.11.

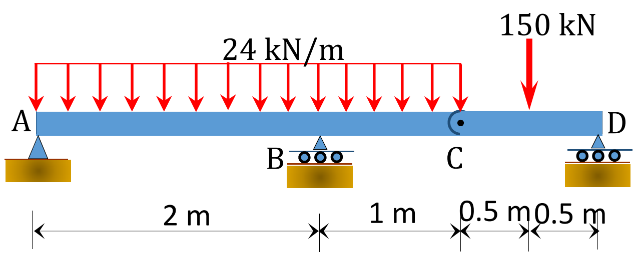


Fig. P4.1. Beam.

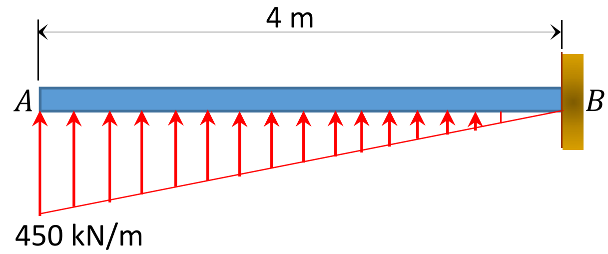


Fig. P4.2. Beam.

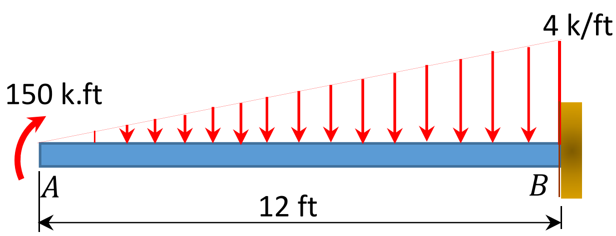


Fig. P4.3. Beam.

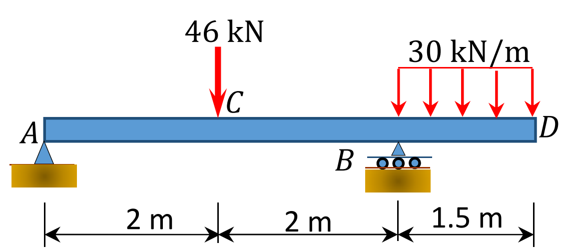


Fig. P4.4. Beam.

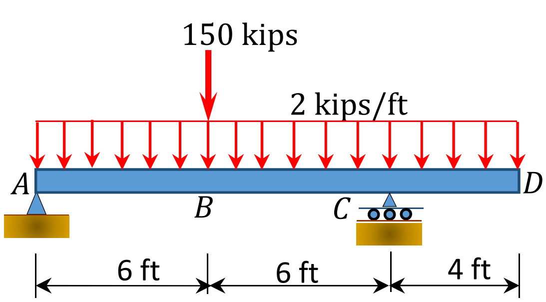


Fig. P4.5. Beam.

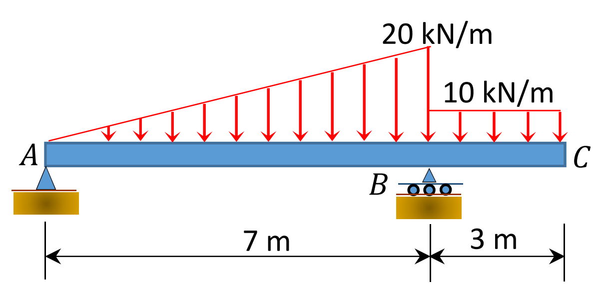


Fig. P4.6. Beam.

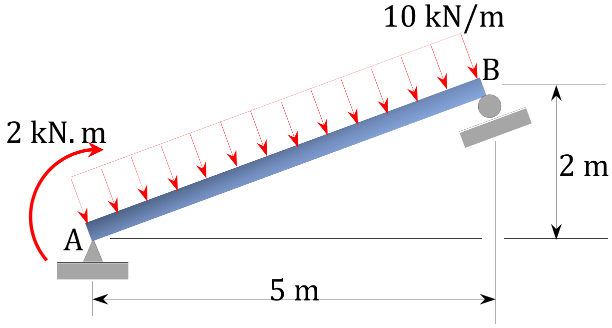


Fig. P4.7. Beam.

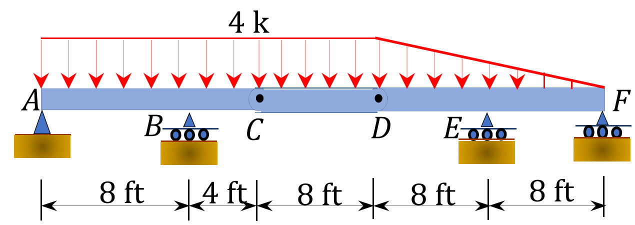


Fig. P4.8. Beam.

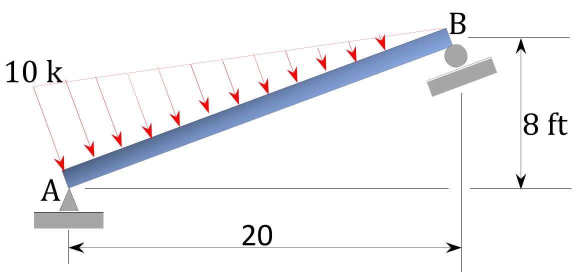


Fig. P4.9. Beam.

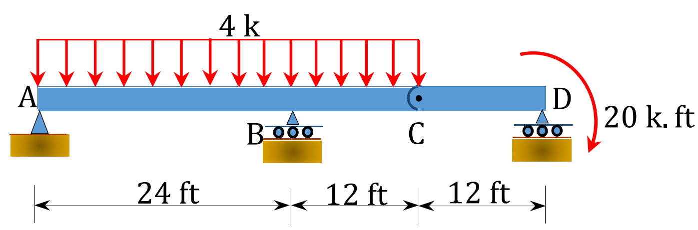


Fig. P4.10. Beam.

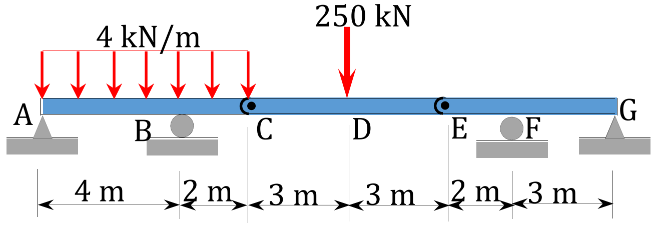


Fig. P4.11. Beam.

4.2. Draw the shearing force and the bending moment diagrams for the frames shown in Figure P4.12 through Figure P4.19.

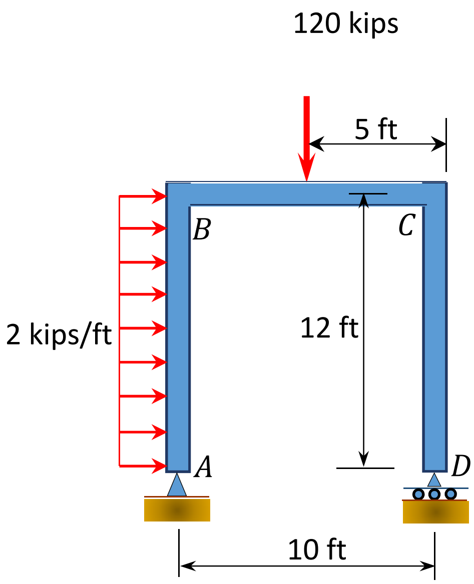


Fig. P4.12. Frame.

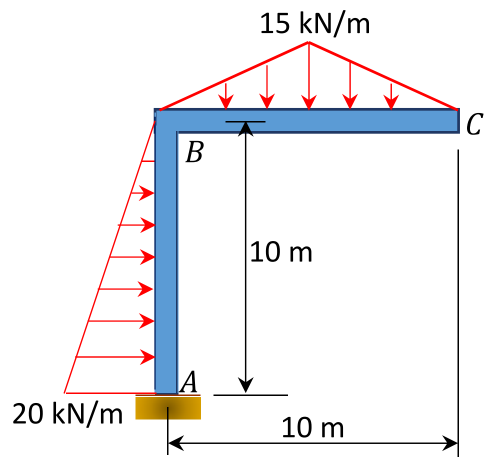


Fig. P4.13. Frame.

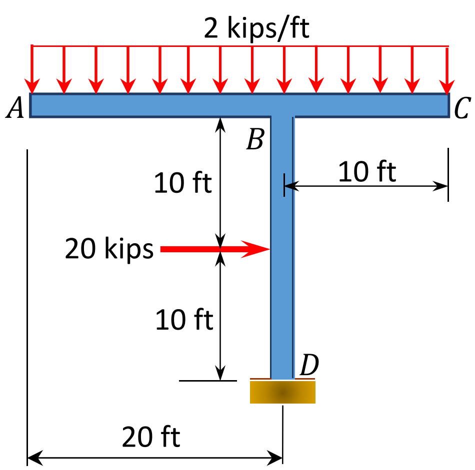


Fig. P4.14. Frame.

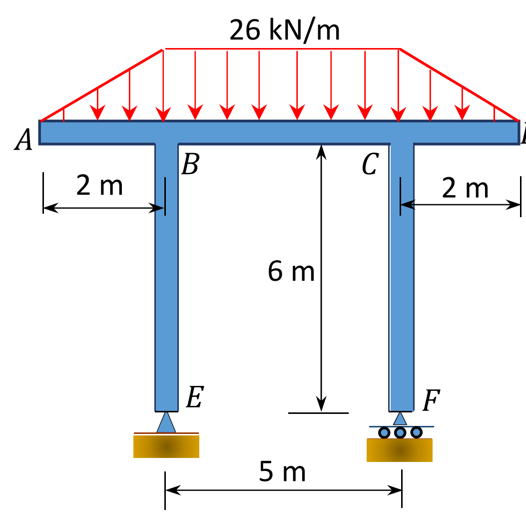


Fig. P4.15. Frame.

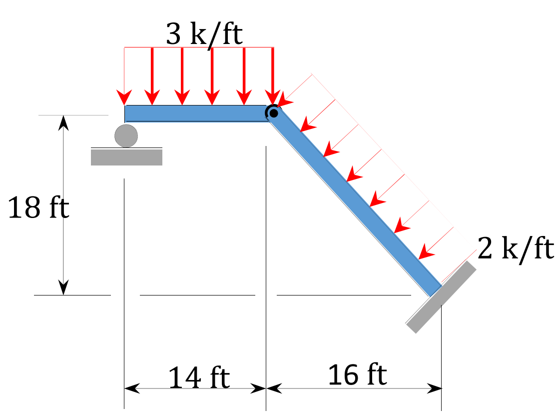


Fig. P4.16. Frame.

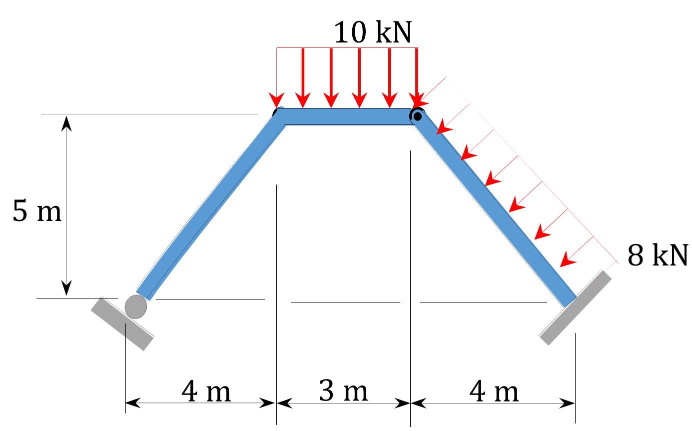


Fig. P4.17. Frame.

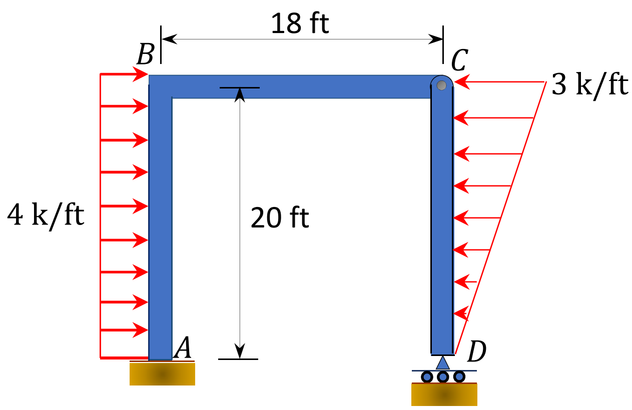


Fig. P4.18. Frame.

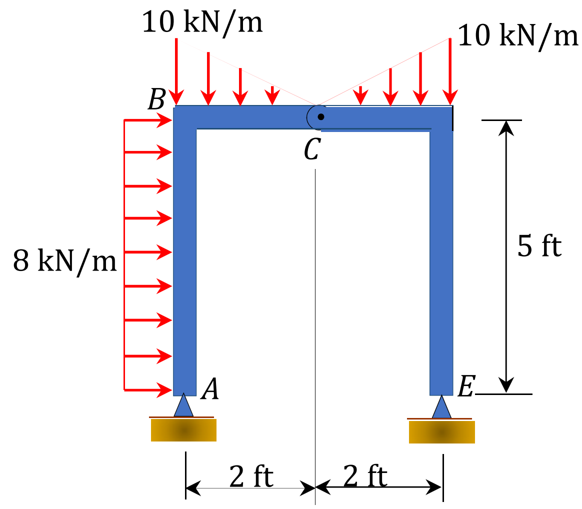


Fig. P4.19. Frame.

Chapter 5

Internal Forces in Plane Trusses

5.1 Introduction

A truss is a structure composed of straight, slender members connected at their ends by frictionless pins or hinges. A truss can be categorized as simple, compound, or complex. A simple truss is one constructed by first arranging three slender members to form a base triangular cell. Additional joints can be formed in the truss by subsequently adding two members at a time to the base cell, as shown in Figure 5.1a. A compound truss consists of two or more simple trusses joined together, as shown in Figure 5.1b. A complex truss is neither simple nor compound, as shown in Figure 5.1c; its analysis is more rigorous than those of the previously stated trusses.

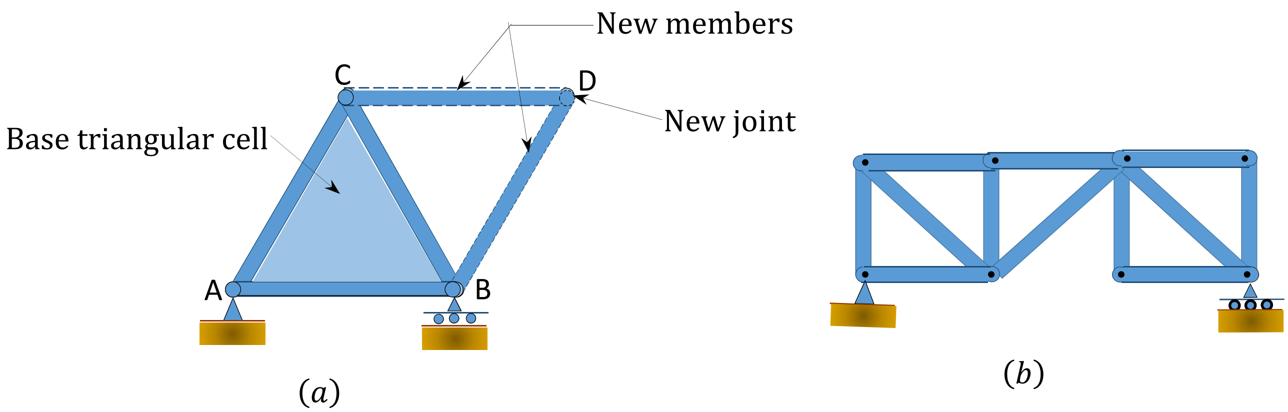
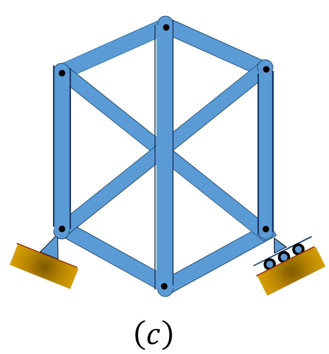


Fig. 5.1. Classification of trusses.



5.2 Types of Trusses

The following are examples of different types of trusses for bridges and roofs.

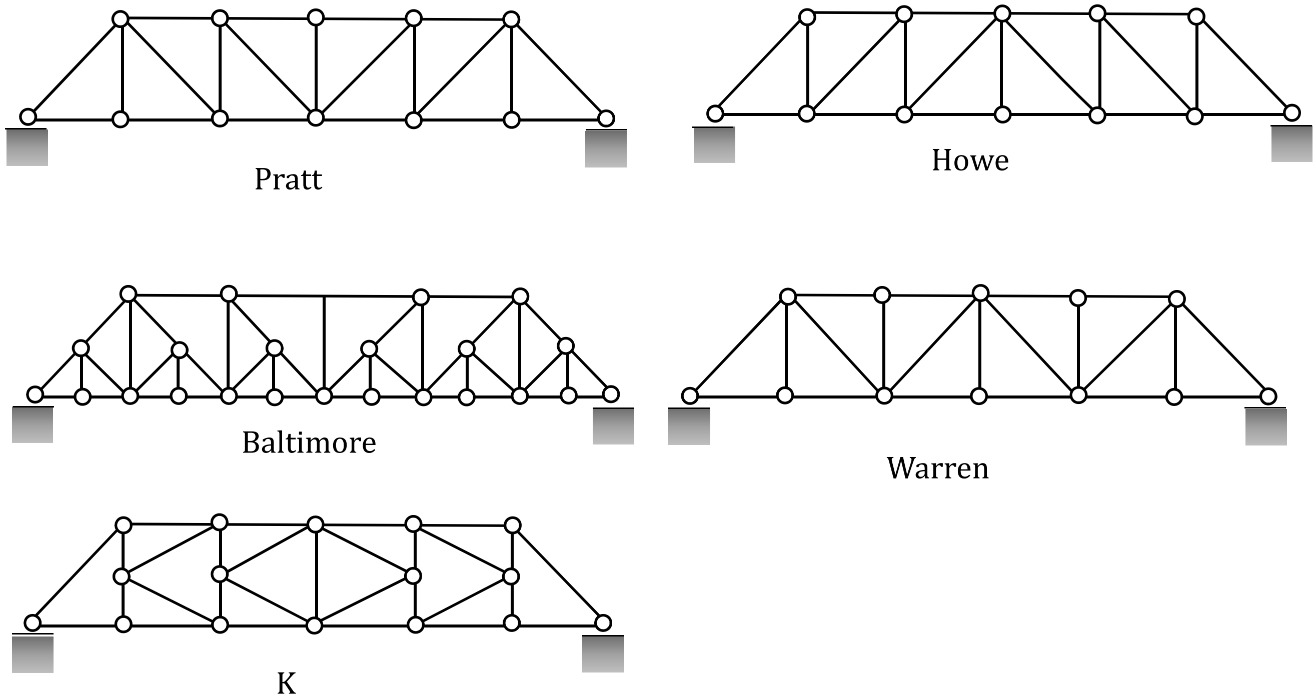


Fig. 5.2. Commonly used bridge trusses.

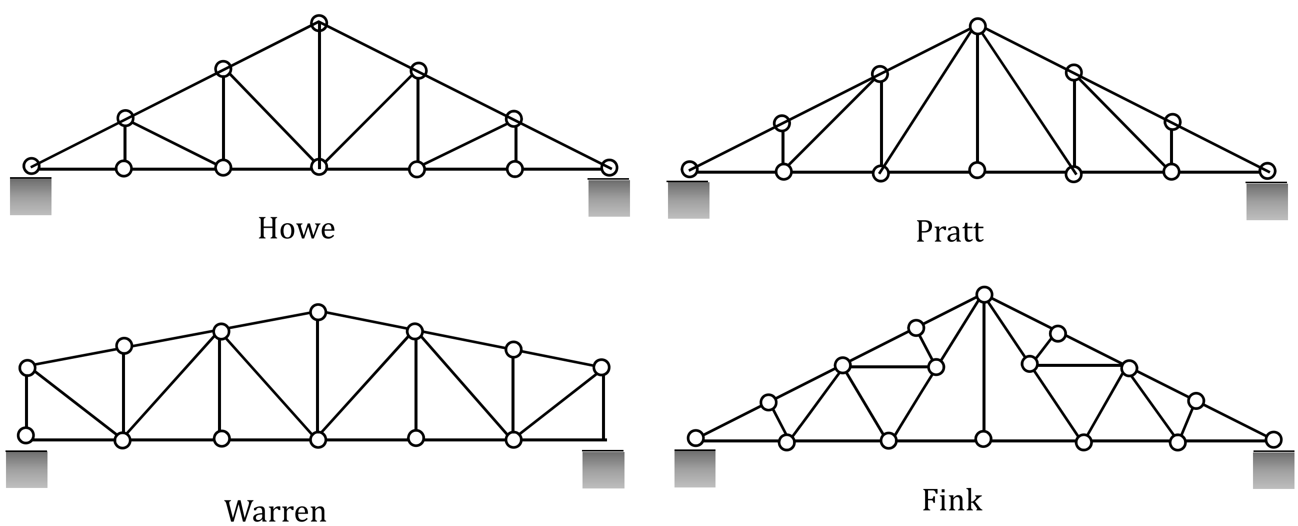


Fig. 5.3. Commonly used roof trusses.

5.3 Determinacy and Stability of Trusses

The conditions of determinacy, indeterminacy, and instability of trusses can be stated as follows:

$$m + r < 2j \quad \text{structure is statically unstable}$$

$$m + r = 2j \quad \text{structure is determinate} \tag{5.1}$$

$$m + r > 2j \quad \text{structure is indeterminate}$$

where

m = number of members.
r = number of support reactions.
j = number of joints.

5.4 Assumptions in Truss Analysis

1. Members are connected at their ends by frictionless pins.
2. Members are straight and, therefore, are subjected only to axial forces.
3. Members' deformation under loads are negligible and of insignificant magnitude to cause appreciable changes in the geometry of the structure.
4. Loads are applied only at the joints due to the arrangement of members.

5.5 Joint Identification and Member Force Notation

Truss joints can be identified using alphabets or numbers, depending on the preference of the analyst. However, consistency must be maintained in the chosen way of identification to avoid confusion during analysis. A bar force can be represented by any letter (F or N or S), with two subscripts designating the member. For example, the member force F_{AB} in the truss shown in Figure 5.4 is the force in the member connecting joints A and B.

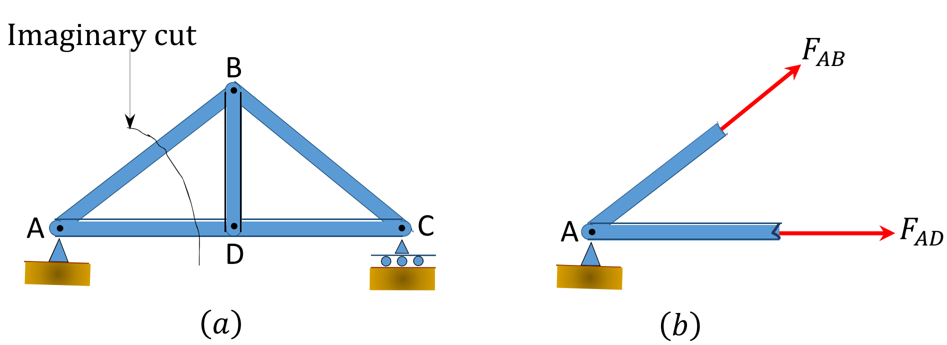
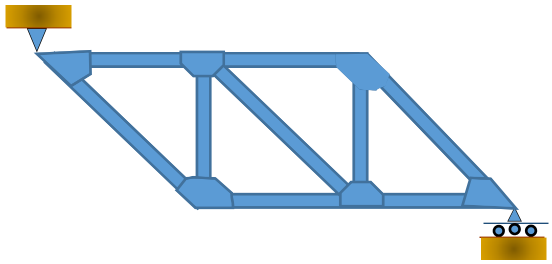


Fig. 5.4. Joint identification (a) and bar force (b).

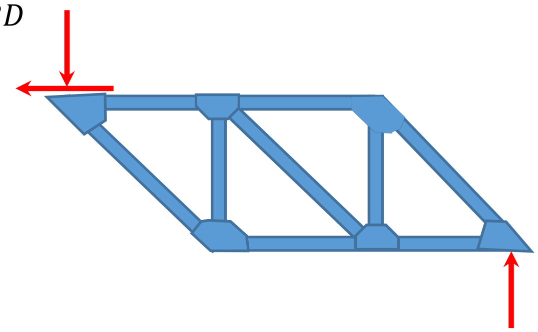
Example 5.1

Classify the trusses shown in Figure 5.5 through Figure 5.9 as stable, determinate, or indeterminate, and state the degree of indeterminacy when necessary.

Fig. 5.5. Truss.

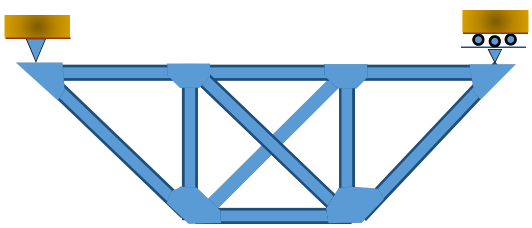


FBD

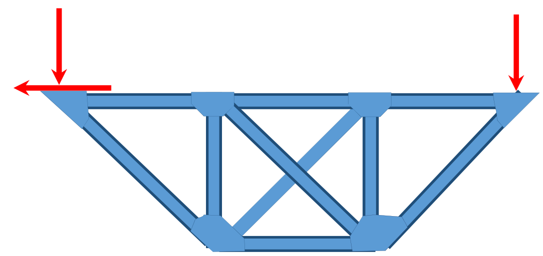


$r = 3, m = 9, j = 6$. From equation 3.5, $9 + 3 = 2(6)$. Statically determinate.

Fig. 5.6. Truss.

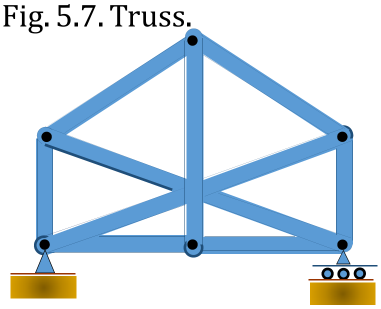


FBD

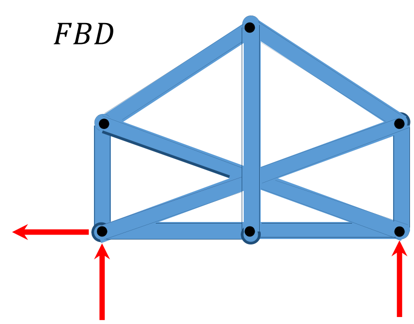


$r = 3, m = 10, j = 6$. From equation 3.5, $10 + 3 > 2(6)$. Statically indeterminate to 1°.

Fig. 5.7. Truss.

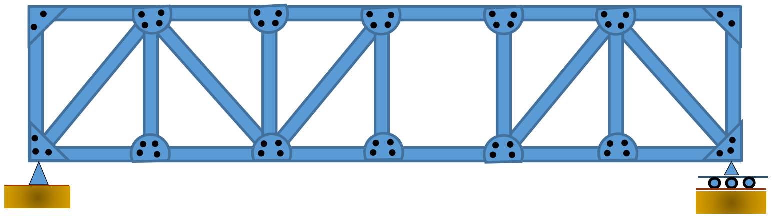


FBD

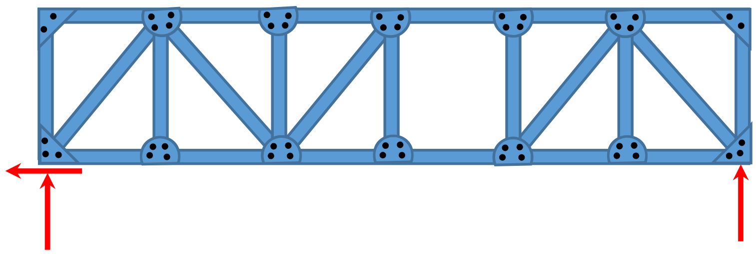


$r = 3, m = 9, j = 6$. From equation 3.5, $9 + 3 = 2(6)$. Statically determinate.

Fig. 5.8. Truss.

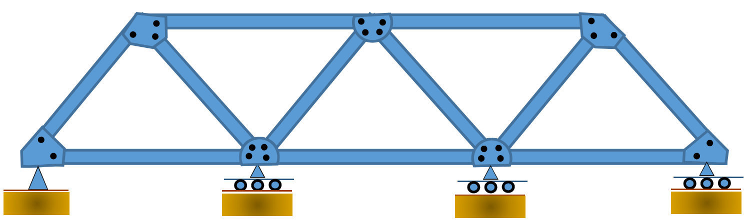


FBD



$r = 3, m = 24, j = 14$. From equation 3.5 , $24 + 3 < 2(14)$. Statically unstable.

Fig. 5.9. Truss.



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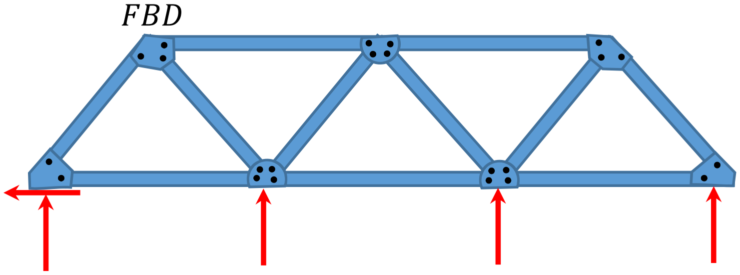


$r = 5, m = 11, j = 7$. From equation 3.5, $11 + 5 > 2(7)$.
Satically indeterminate to the 2°.

5.6 Methods of Truss Analysis

There are several methods of truss analysis, but the two most common are the method of joint and the method of section (or moment).

5.6.1 Sign Convention

In truss analysis, a negative member axial force implies that the member or the joints at both ends of the member are in compression, while a positive member axial force indicates that the member or the joints at both ends of the member are in tension.

5.6.2 Analysis of Trusses by Method of Joint

This method is based on the principle that if a structural system constitutes a body in equilibrium, then any joint in that system is also in equilibrium and, thus, can be isolated from the entire system and analyzed using the conditions of equilibrium. The method of joint involves successively isolating each joint in a truss system and determining the axial forces in the members meeting at the joint by applying the equations of equilibrium. The detailed procedure for analysis by this method is stated below.

Procedure for Analysis

- Verify the stability and determinacy of the structure. If the truss is stable and determinate, then proceed to the next step.
- Determine the support reactions in the truss.
- Identify the zero-force members in the system. This will immeasurably reduce the computational efforts involved in the analysis.
- Select a joint to analyze. At no instance should there be more than two unknown member forces in the analyzed joint.
- Draw the isolated free-body diagram of the selected joint, and indicate the axial forces in all members meeting at the joint as tensile (i.e. as pulling away from the joint). If this initial assumption is wrong, the determined member axial force will be negative in the analysis, meaning that the member is in compression and not in tension.
- Apply the two equations $\sum F_x = 0$ and $\sum F_y = 0$ to determine the member axial forces.
- Continue the analysis by proceeding to the next joint with two or fewer unknown member forces.

Example 5.2

Using the method of joint, determine the axial force in each member of the truss shown in Figure 5.10a.

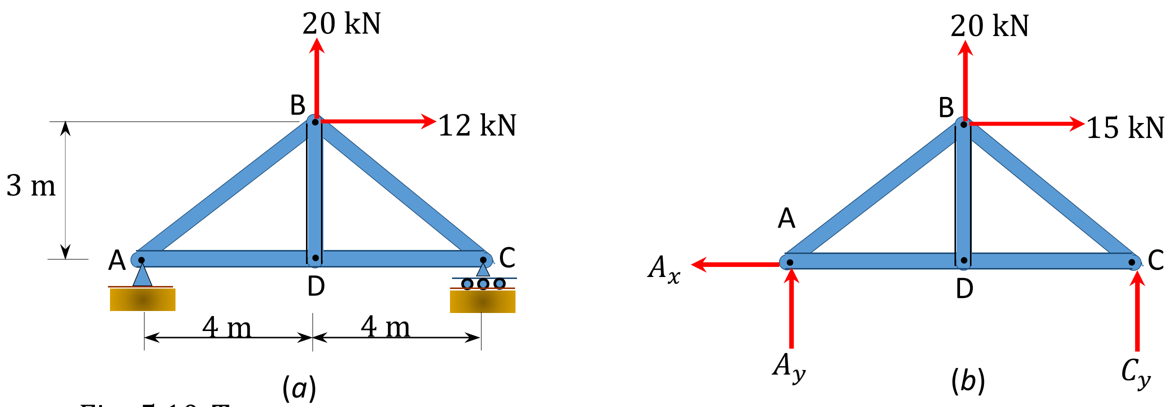


Fig. 5.10. Truss.

Solution

Support reactions. By applying the equations of static equilibrium to the free-body diagram shown in Figure 5.10b, the support reactions can be determined as follows:

$$+\curvearrowleft \sum M_A = 0$$

$$20(4) - 12(3) + (8)C_y = 0$$

$$C_y = -5.5 \text{ kN} \qquad C_y = 5.5 \text{ kN} \downarrow$$

$$+\uparrow \sum F_y = 0$$

$$A_y - 5.5 + 20 = 0$$

$$A_y = -14.5 \text{ kN} \qquad A_y = 14.5 \text{ kN} \downarrow$$

$$+\rightarrow \sum F_x = 0$$

$$-A_x + 12 = 0$$

$$A_x = 12 \text{ kN} \qquad A_x = 12 \text{ kN} \leftarrow$$

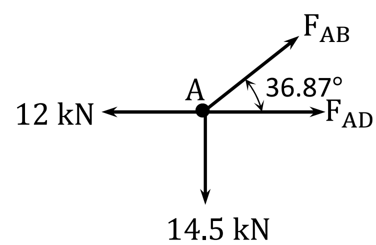
Analysis of joints. The analysis begins with selecting a joint that has two or fewer unknown member forces. The free-body diagram of the truss will show that joints A and B satisfy this requirement. To determine the axial forces in members meeting at joint A, first isolate the joint from the truss and indicate the axial forces of members as F_{AB} and $F_{AD,}$ as shown in Figure 5.10c. The two unknown forces are initially assumed to be tensile (i.e. pulling away from the joint). If this initial assumption is incorrect, the computed values of the axial forces will be negative, signifying compression.

Analysis of joint *A*.

$+\uparrow \sum F_y = 0$
$F_{AB} sin 36.87° - 14.5 = 0$
$F_{AB} = 24.17$

$+\rightarrow \sum F_x = 0$
$-12 + F_{AD} + F_{AB} \cos 36.87° = 0$

$F_{AD} = 12 - 24.17 \cos 36.87° = -7.34 \text{ kN}$



(c)

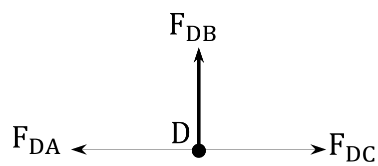
After completing the analysis of joint *A,* joint *B* or *D* can be analyzed, as there are only two unknown forces.

Analysis of joint *D*.

$+\uparrow \sum F_y = 0$

$F_{DB} = 0$

$+\rightarrow \sum F_x = 0$
$-F_{DA} + F_{DC} = 0$
$F_{DC} = F_{DA} = -7.34 \text{ kN}$

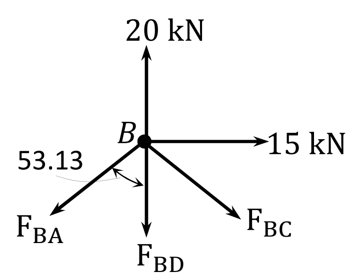


(d)

Analysis of joint *B*.

$+\rightarrow \sum F_x = 0$
$-F_{BA} sin\ 53.13 + F_{BC} sin\ 53.13 + 15 = 0$
$F_{BC} sin 53.13 = -15 + 24.17 sin 53.13 =$

$F_{BC} = 5.42 \text{ kN}$



(e)

5.6.3 Zero Force Members

Complex truss analysis can be greatly simplified by first identifying the “zero force members.” A zero force member is one that is not subjected to any axial load. Sometimes, such members are introduced into the truss system to prevent the buckling and vibration of other members. The truss-member arrangements that result in zero force members are listed as follows:

1. If noncollinearity exists between two members meeting at a joint that is not subjected to any external force, then the two members are zero force members (see Figure 5.11a).
2. If three members meet at a joint with no external force, and two of the members are collinear, the third member is a zero force member (see Figure 5.11b).
3. If two members meet at a joint, and an applied force at the joint is parallel to one member and perpendicular to the other, then the member perpendicular to the applied force is a zero force member (see Figure 5.11c).

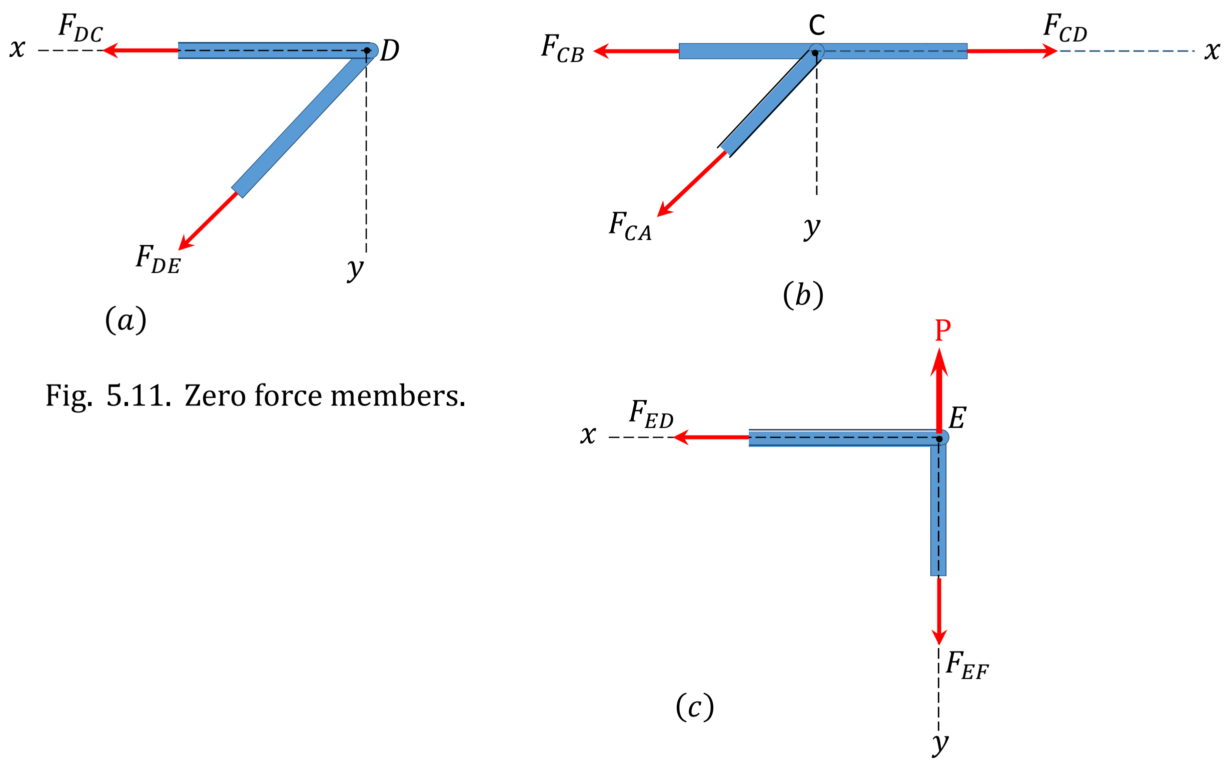


Fig. 5.11. Zero force members.

5.6.4 Analysis of Trusses by Method of Section

Sometimes, determining the axial force in specific members of a truss system by the method of joint can be very involving and cumbersome, especially when the system consists of several members. In such instances, using the method of section can be timesaving and, thus, preferable. This method involves passing an imaginary section through the truss so that it divides the system into two parts and cuts through members whose axial forces are desired. Member axial forces are then determined using the conditions of equilibrium. The detailed procedure for analysis by this method is presented below.

Procedure for Analysis of Trusses by Method of Section

- Check the stability and determinacy of the structure. If the truss is stable and determinate, then proceed to the next step.
- Determine the support reactions in the truss.
- Make an imaginary cut through the structure so that it includes the members whose axial forces are desired. The imaginary cut divides the truss into two parts.
- Apply forces to each part of the truss to keep it in equilibrium.
- Select either part of the truss for the determination of member forces.
- Apply the conditions of equilibrium to determine the member axial forces.

Example 5.3

Using the method of section, determine the axial forces in members *CD*, *CG*, and *HG* of the truss shown in Figure 5.12a.

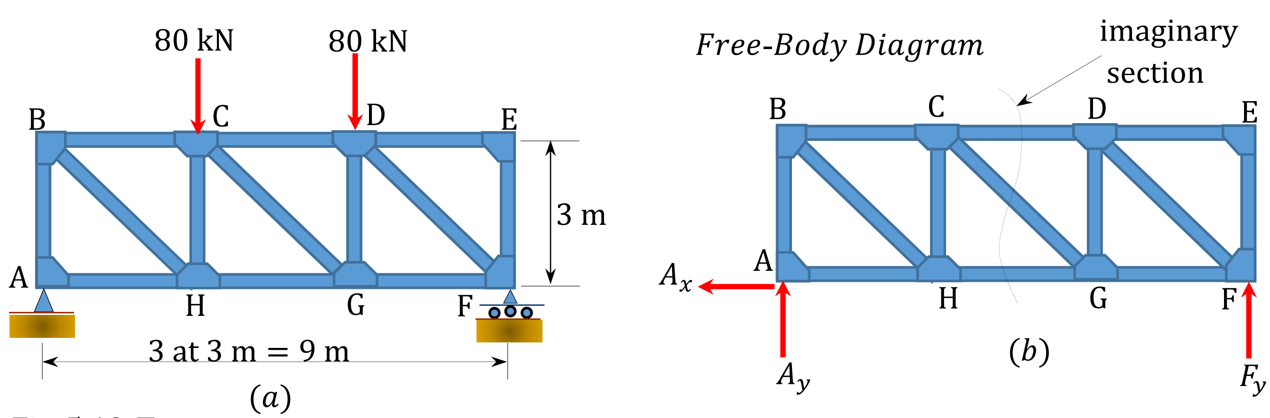
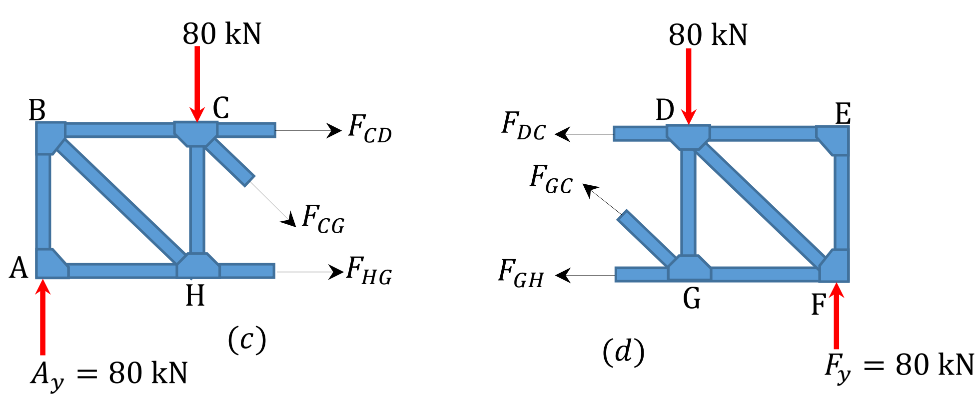


Fig. 5.12. Truss.



Solution

Support reactions. By applying the equations of static equilibrium to the free-body diagram in Figure 5.12b, the support reactions can be determined as follows:

$A_y = F_y = \frac{160}{2} = 80 \text{ kN}$
$+\rightarrow \sum F_x = 0 \quad A_x = 0$

Analysis by method of section. First, an imaginary section is passed through the truss so that it cuts through members *CD*, *CG*, and *HG* and divides the truss into two parts, as shown in Figure 5.12c and Figure 5.12d. Member forces are all indicated as tensile forces (i.e., pulling away from the joint). If this initial assumption is wrong, the calculated member forces will be negative, showing that they are in compression. Either of the two parts can be used for the analysis. The left-hand part will be used for determining the member forces in this example. By applying the equation of equilibrium to the left-hand segment of the truss, the axial forces in members can be determined as follows:

Axial force in member *CD*. To determine the axial force in member *CD,* find a moment about a joint in the truss where only *CD* will have a moment about that joint and all other cut members will have no moment. A close examination will show that the joint that meets this requirement is joint *G*. Thus, taking the moment about *G* suggests the following:

$+\circlearrowleft \sum M_G = 0$
$-80(6) + 80(3) - F_{CD}(3) = 0$
$F_{CD} = -80 \, kN \qquad\qquad 80 \text{ kN}(C)$

Axial force in member *HG*.

$+\circlearrowleft \sum M_C = 0$
$-80(3) + F_{HG}(3) = 0$
$F_{HG} = 80 \text{ kN} \qquad\qquad 80 \text{ kN}(T)$

Axial force in member *CG*. The axial force in member *CG* is determined by considering the vertical equilibrium of the left-hand part. Thus,

$+\uparrow \sum F_y = 0$

$80 - 80 - F_{CG} \cos 45° = 0$
$F_{CG} = 0$

Chapter Summary

Internal forces in plane trusses: Trusses are structural systems that consist of straight and slender members connected at their ends. The assumptions in the analysis of plane trusses include the following:

1. Members of trusses are connected at their ends by frictionless pins.
2. Members are straight and are subjected to axial forces.
3. Members' deformations are small and negligible.
4. Loads in trusses are only applied at their joints.

Members of a truss can be subjected to axial compression or axial tension. Axial compression of members is always considered negative, while axial tension is always considered positive.

Trusses can be externally or internally determinate or indeterminate. Externally determinate trusses are those whose unknown external reactions can be determined using only the equation of static equilibrium. Externally indeterminate trusses are those whose external unknown reaction cannot be determined completely using the equations of equilibrium. To determine the number of unknown reactions in excess of the equation of equilibrium for the indeterminate trusses, additional equations must be formulated based on the compatibility of parts of the system. Internally determinate trusses are those whose members are so arranged that just enough triangular cells are formed to prevent geometrical instability of the system.

The formulation of stability and determinacy in trusses is as follows:

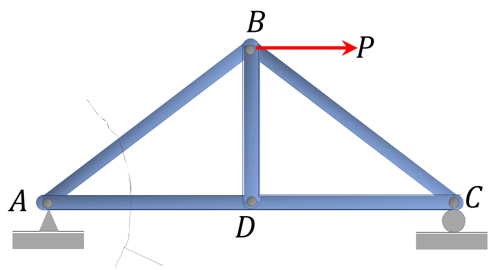
$m + r < 2j$ Structure is unstable

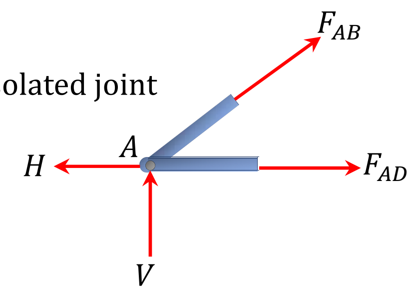
$m + r = 2j$ Structure is determinate

$m + r > 2j$ Structure is indeterminate

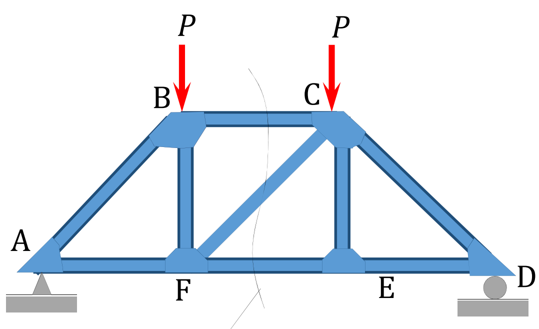
Methods of analysis of trusses: The two common methods of analysis of trusses are the method of joint and the method of section (or moment).

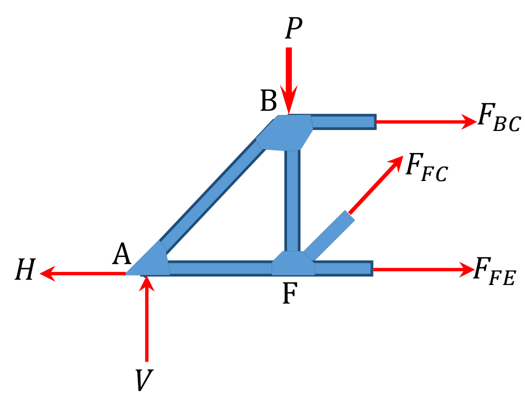
Method of joint: This method involves isolating each joint of the truss and considering the equilibrium of the joint when determining the member axial force. Two equations used in determining the member axial forces are $\sum F_x = 0$ and $\sum F_y = 0$. Joints are isolated consecutively for analysis based on the principle that the number of the unknown member axial forces should never be more than two in the joint under consideration in a plane trust.





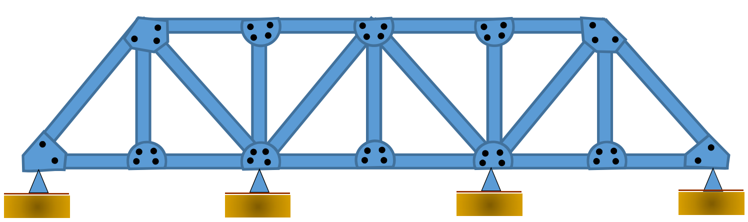
Method of section: This method entails passing an imaginary section through the truss to divide it into two sections. The member forces are determined by considering the equilibrium of the part of the truss on either side of the section. This method is advantageous when the axial forces in specific members are required in a truss with several members.



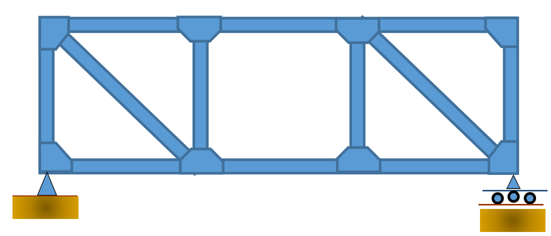


Practice Problems

5.1 Classify the trusses shown in Figure P5.1a through Figure P5.1r.



(a)



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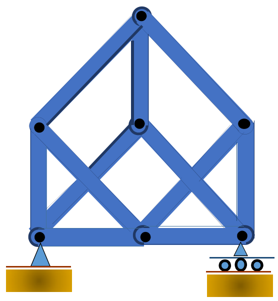

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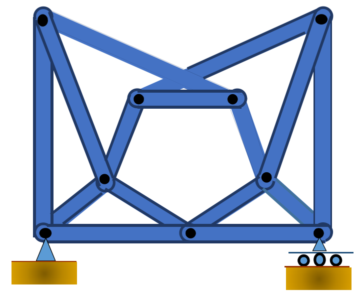

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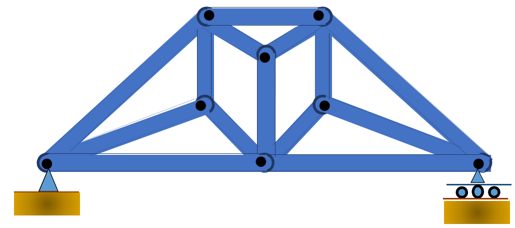

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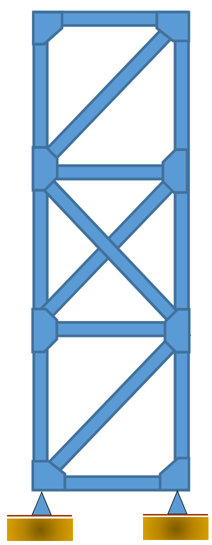

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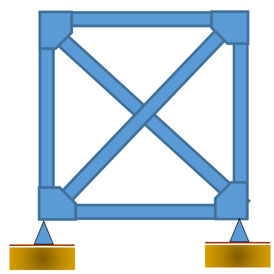

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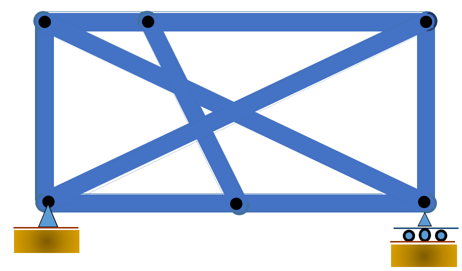

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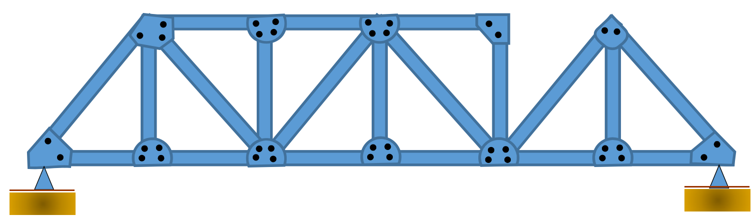
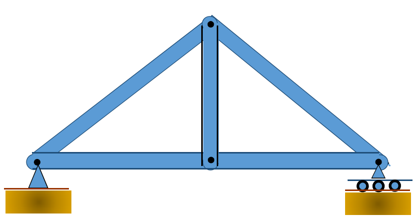
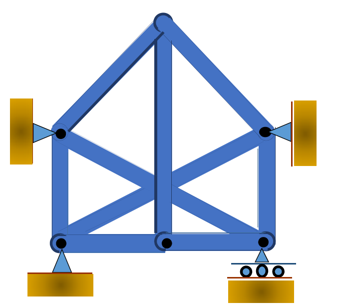


(i)



(j)



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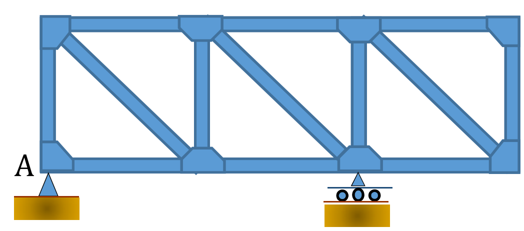

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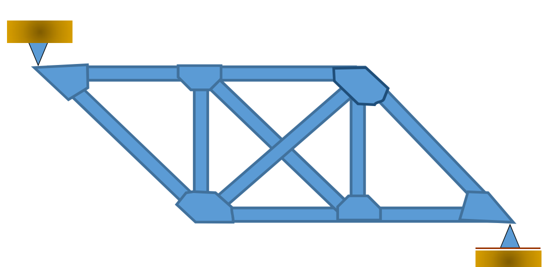

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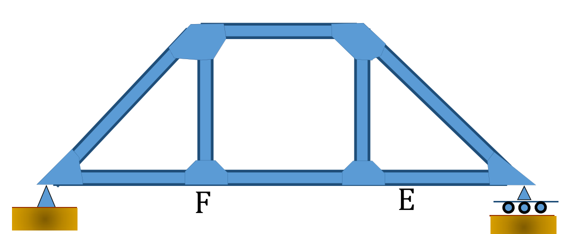

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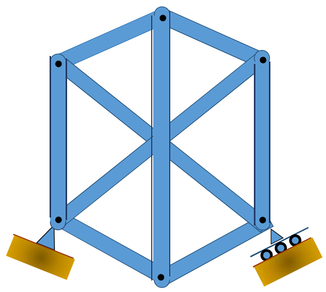

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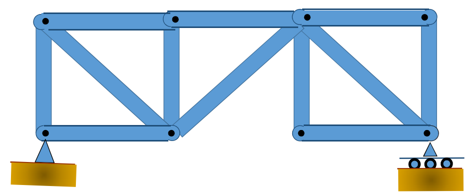

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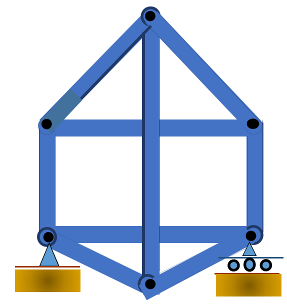

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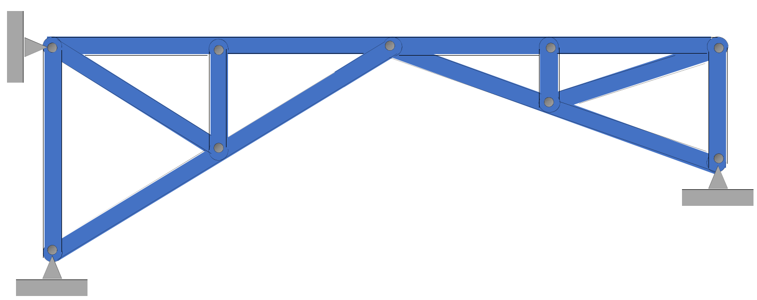


(r)

P5.1. Truss classification.

5.2 Determine the force in each member of the trusses shown in Figure P5.2 through Figure P5.12 using the method of joint.

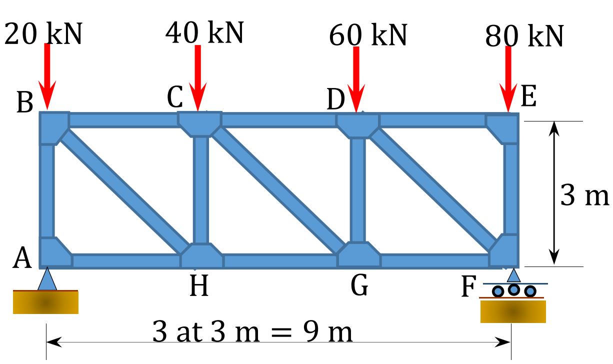


Fig. P5.2. Truss.

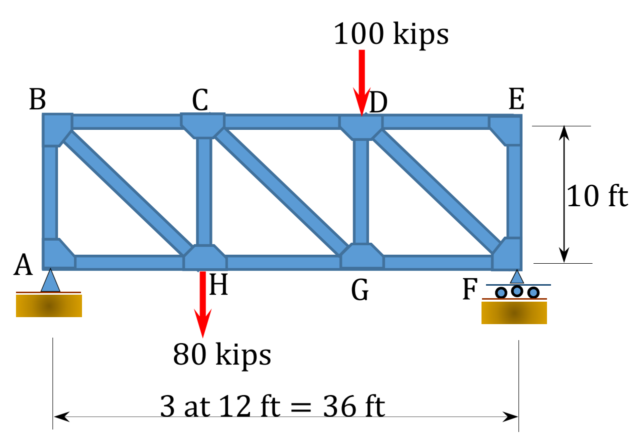


Fig. P5.3. Truss.

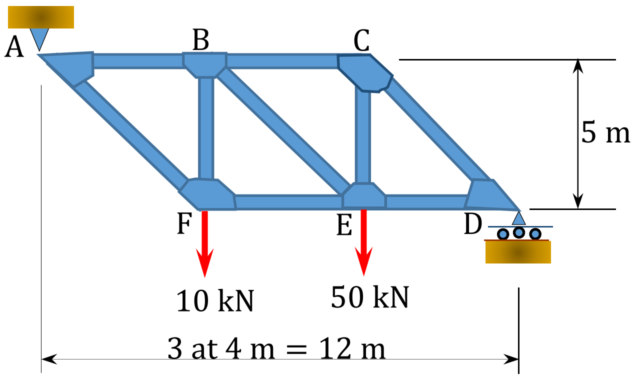


Fig. P5.4. Truss.

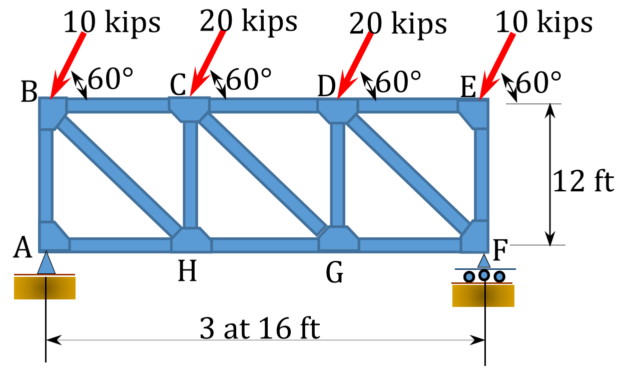


Fig. P5.5. Truss.

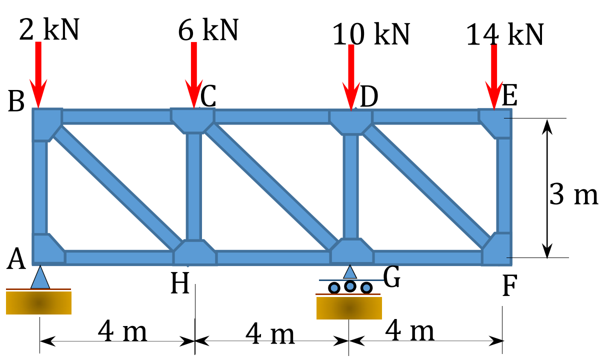


Fig. P5.6. Truss.

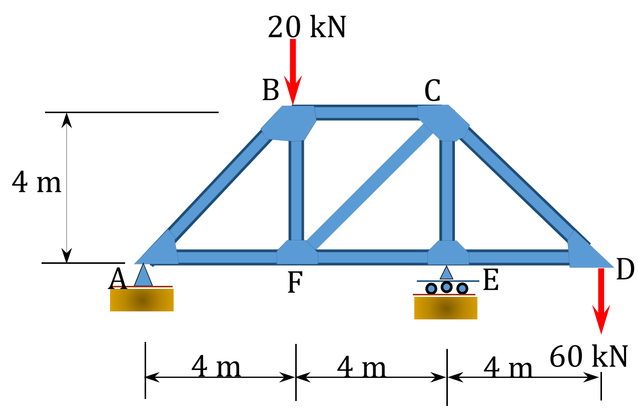


Fig. P5.7. Truss.

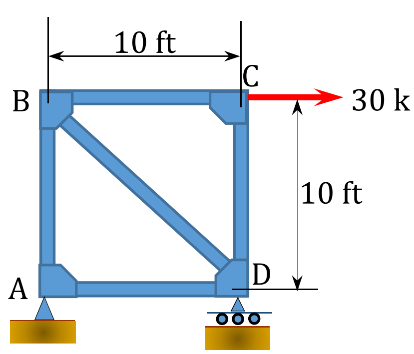


Fig. P5.8. Truss.

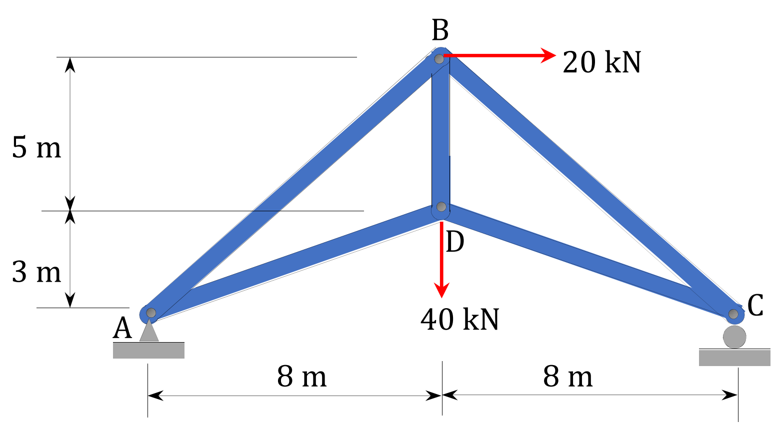


Fig. P5.9. Truss.

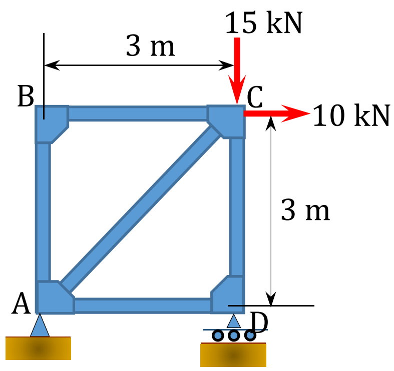


Fig. P5.10. Truss.

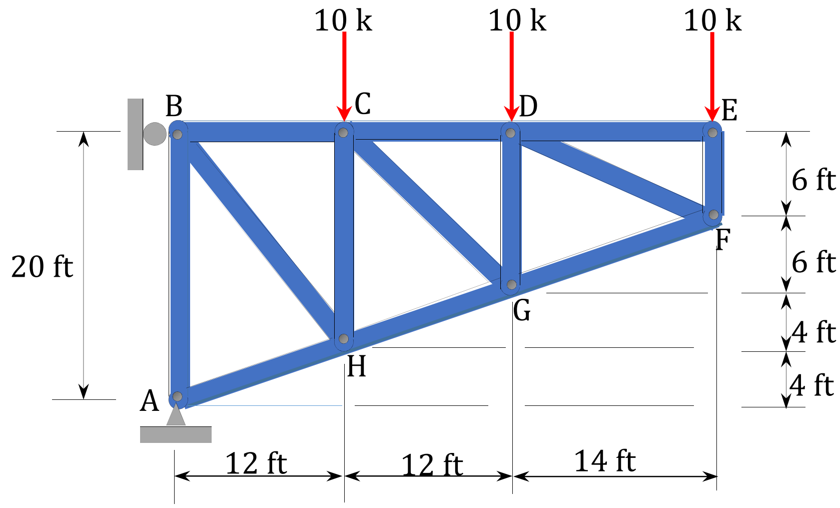


Fig. P5.11. Truss.

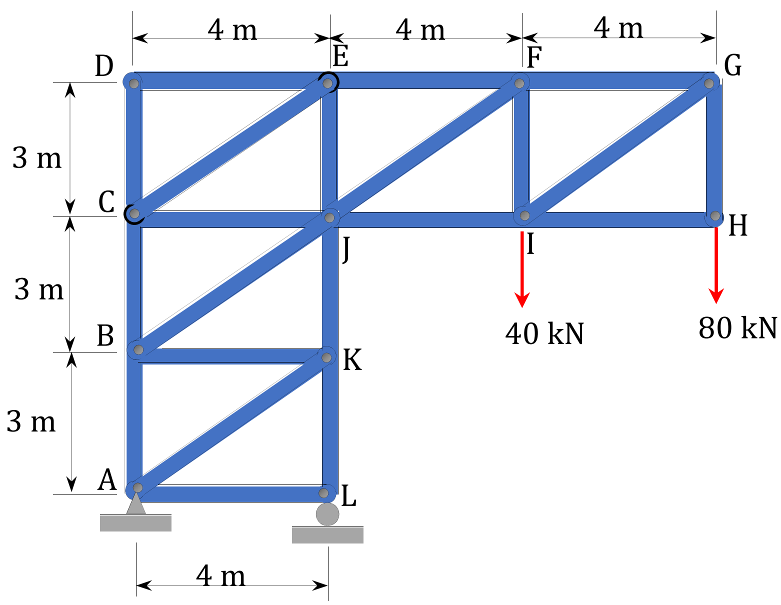


Fig. 5.12. Truss.

5.3 Using the method of section, determine the forces in the members marked X of the trusses shown in Figure P5.13 through Figure P5.19.

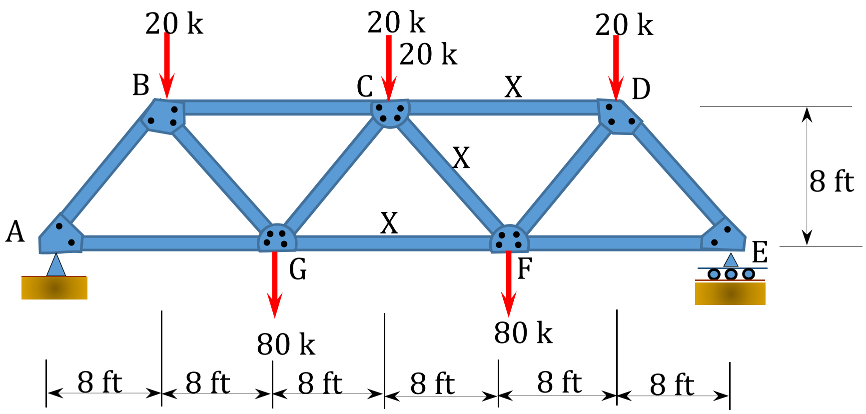


Fig. P5.13. Truss.

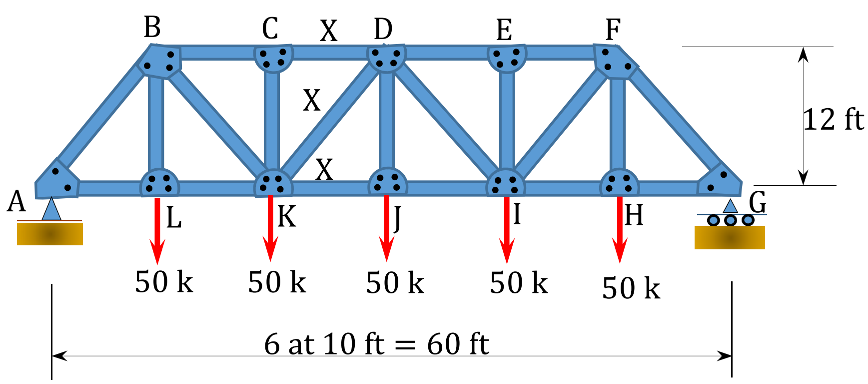


Fig. P5.14. Truss.

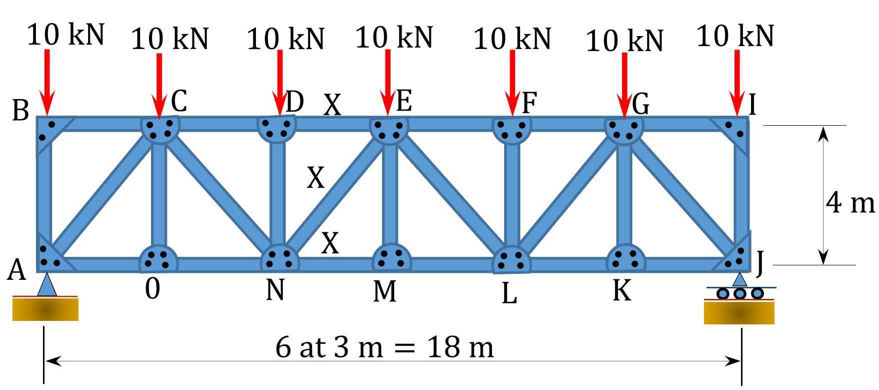


Fig. P5.15. Truss.

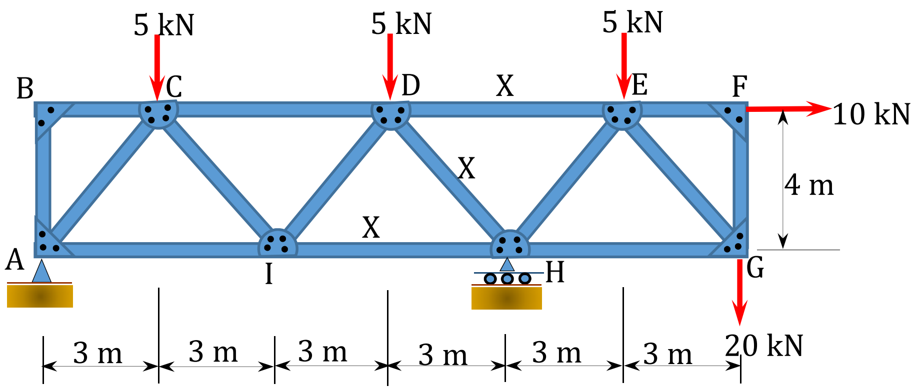


Fig. P5.16. Truss.

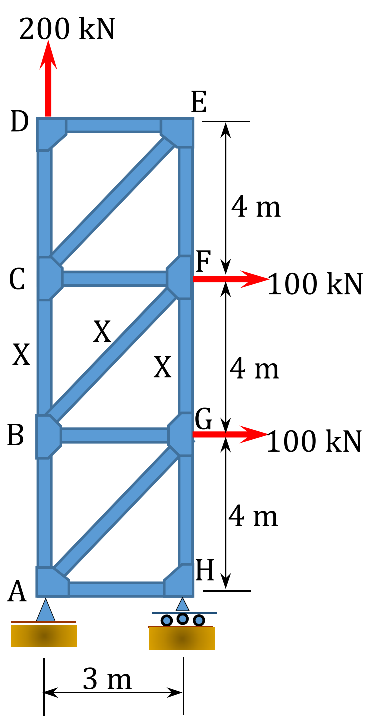


Fig. P5.17. Truss.

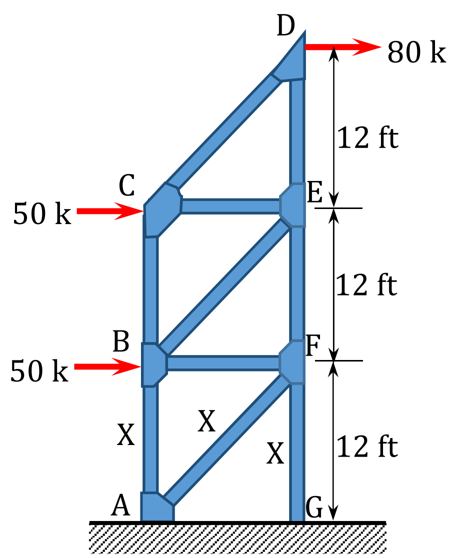


Fig. P5.18. Truss.

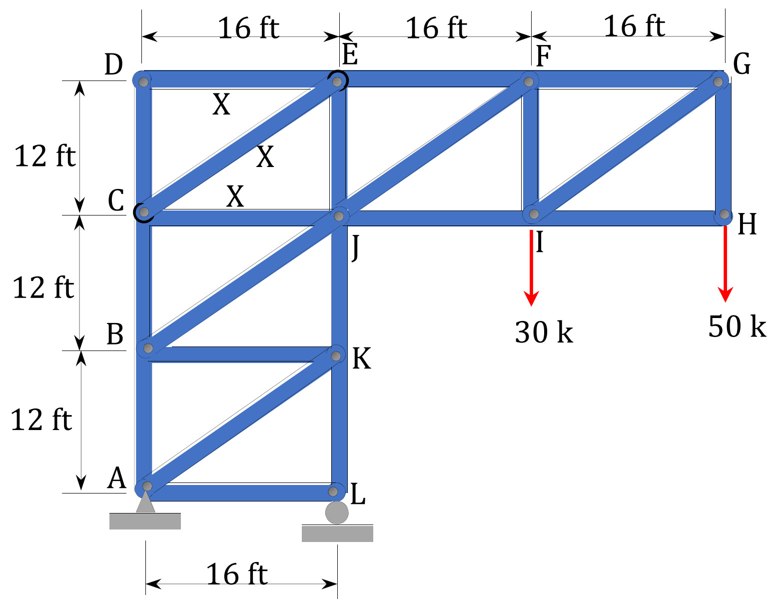


Fig. P5.19. Truss.

Chapter 6

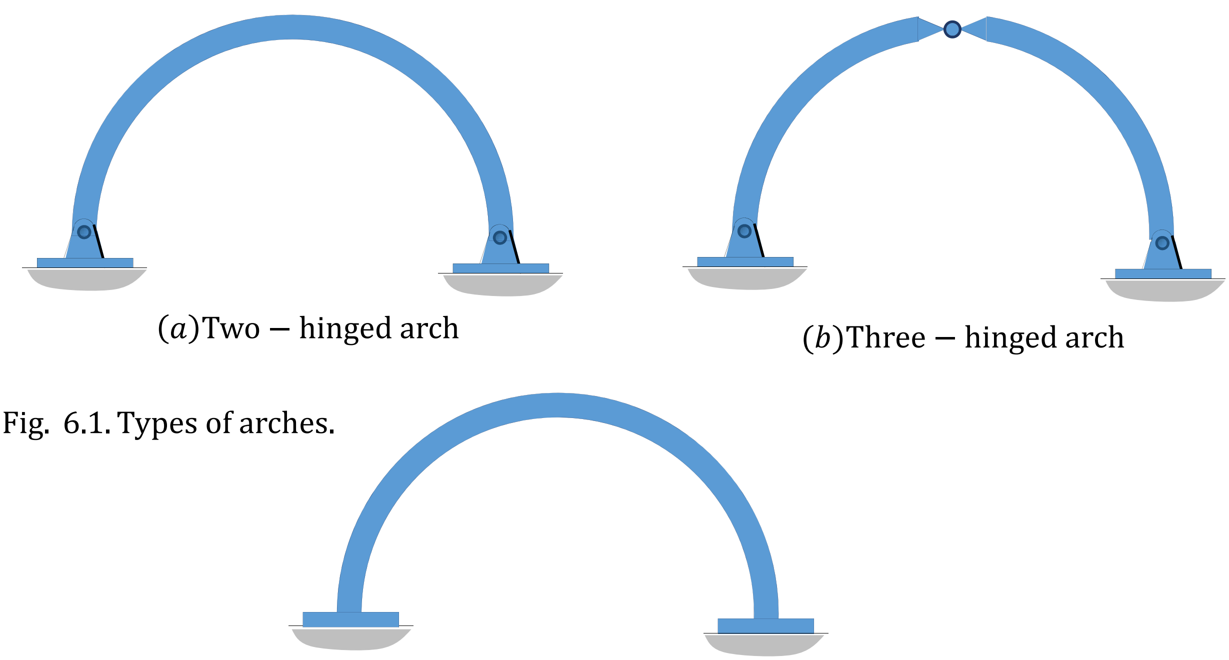
Arches and Cables

6.1 Arches

Arches are structures composed of curvilinear members resting on supports. They are used for large-span structures, such as airplane hangars and long-span bridges. One of the main distinguishing features of an arch is the development of horizontal thrusts at the supports as well as the vertical reactions, even in the absence of a horizontal load. The internal forces at any section of an arch include axial compression, shearing force, and bending moment. The bending moment and shearing force at such section of an arch are comparatively smaller than those of a beam of the same span due to the presence of the horizontal thrusts. The horizontal thrusts significantly reduce the moments and shear forces at any section of the arch, which results in reduced member size and a more economical design compared to other structures. Additionally, arches are also aesthetically more pleasant than most structures.

6.1.1 Types of Arches

Based on their geometry, arches can be classified as semicircular, segmental, or pointed. Based on the number of internal hinges, they can be further classified as two-hinged arches, three-hinged arches, or fixed arches, as seen in Figure 6.1. This chapter discusses the analysis of three-hinge arches only.



(a)Two $-$ hinged arch

(b)Three $-$ hinged arch

(c) Fixed arch

Fig. 6.1. Types of arches.

6.1.2 Three-Hinged Arch

A three-hinged arch is a geometrically stable and statically determinate structure. It consists of two curved members connected by an internal hinge at the crown and is supported by two hinges at its base. Sometimes, a tie is provided at the support level or at an elevated position in the arch to increase the stability of the structure.

6.1.2.1 Derivation of Equations for the Determination of Internal Forces in a Three-Hinged Arch

Consider the section Q in the three-hinged arch shown in Figure 6.2a. The three internal forces at the section are the axial force, N_Q, the radial shear force, V_Q, and the bending moment, M_Q. The derivation of the equations for the determination of these forces with respect to the angle φ are as follows:

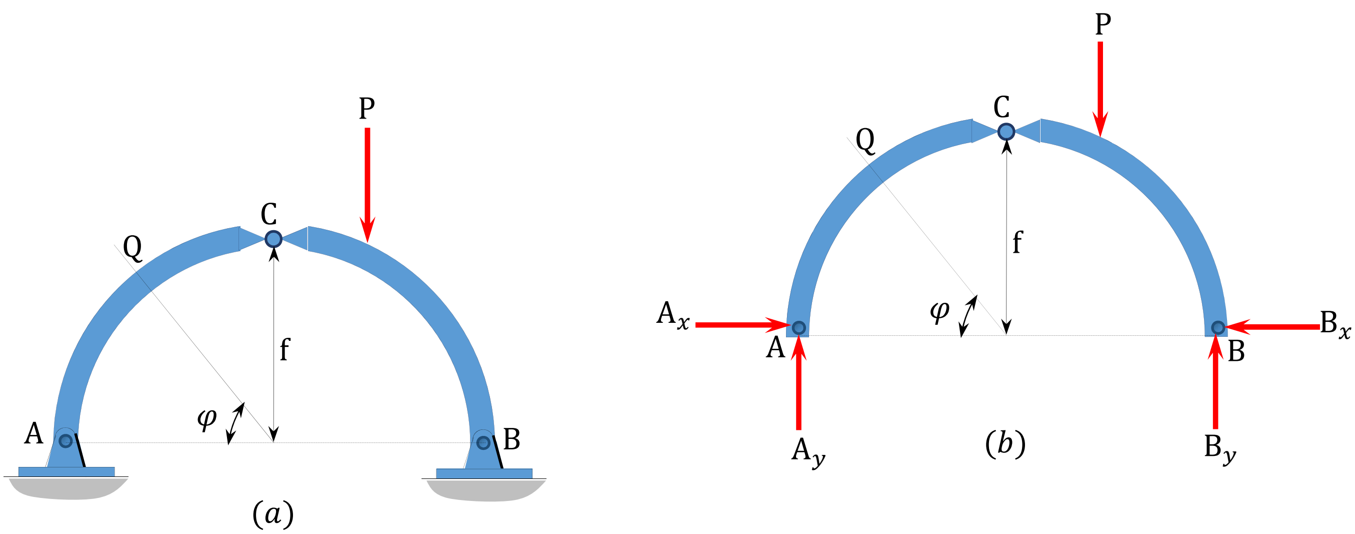
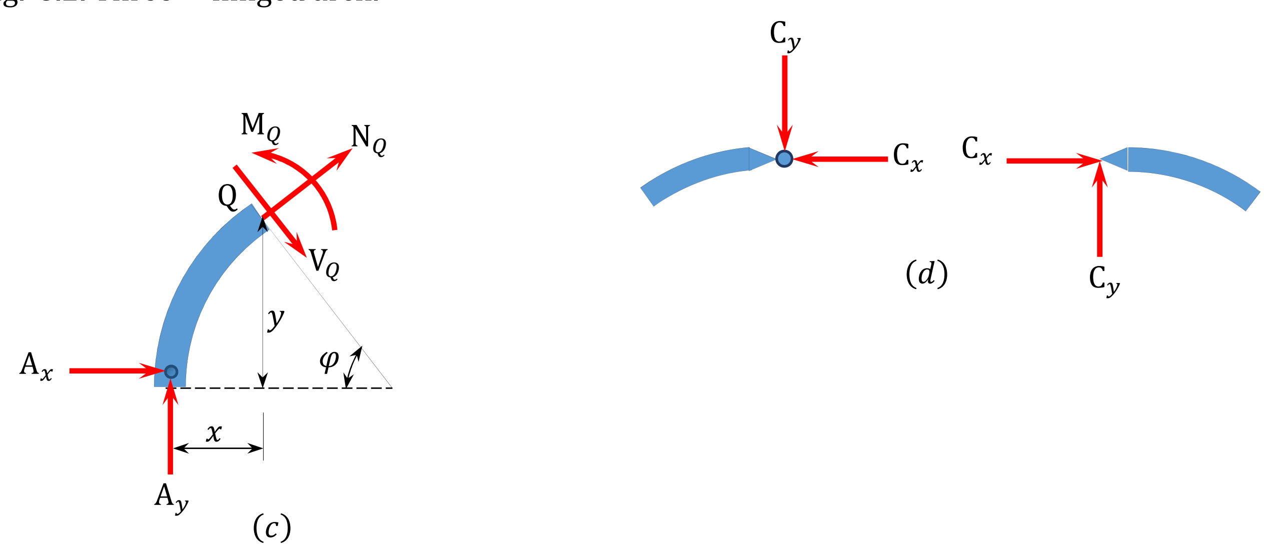


Fig. 6.2. Three – hinged arch.



Bending moment at point Q.

$$M_\varphi = A_y x - A_x y = M^b_{(x)} - A_x y \qquad (6.1)$$

where

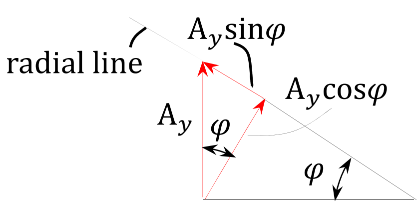
$M^b_{(x)}$ = moment of a beam of the same span as the arch.
y = ordinate of any point along the central line of the arch.

For a parabolic arch, $y = \frac{4fx}{L^2}(L - x)$ (6.2)

For a circular arch, $y = \sqrt{R^2 - \left(\frac{L}{2} - x\right)^2}\, R + f$ (6.3)

f = rise of arch. This is the vertical distance from the centerline to the arch's crown.
x = horizontal distance from the support to the section being considered.
L = span of arch.
R = radius of the arch's curvature.

Radial shear force at point Q.



(e)

$$V_\varphi = A_y \sin\varphi - A_x \cos\varphi = V^b \sin\varphi - A_x \cos\varphi \qquad (6.4)$$

where

V^b = shear of a beam of the same span as the arch.

Axial force at a point Q.

$$N_\varphi = -A_y \cos\varphi - A_x \sin\varphi = -V^b \cos\varphi - A_x \sin\varphi \qquad (6.5)$$

Example 6.1

A three-hinged arch is subjected to two concentrated loads, as shown in Figure 6.3a. Determine the support reactions of the arch.

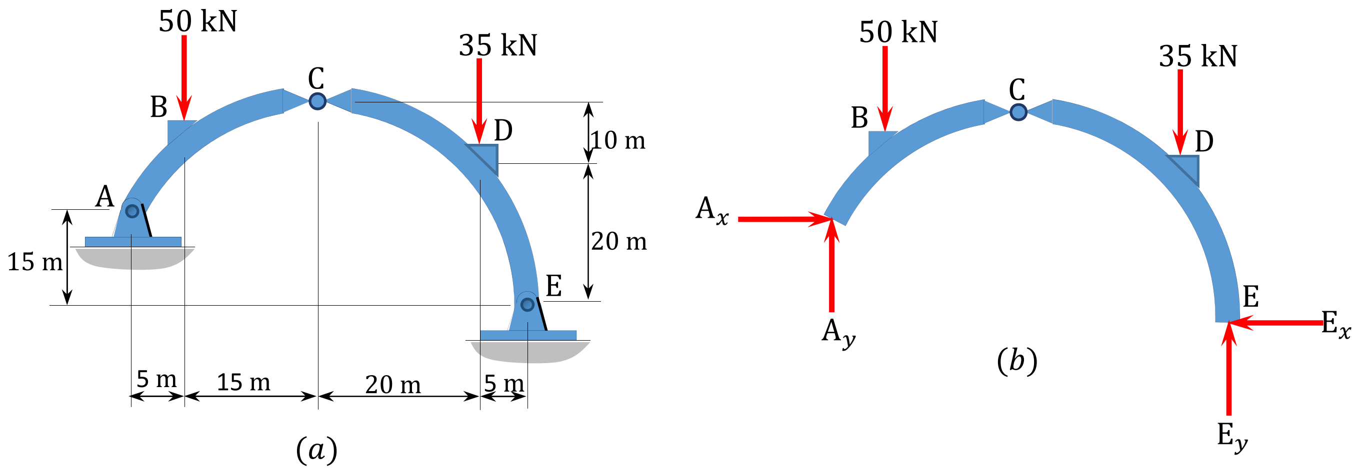
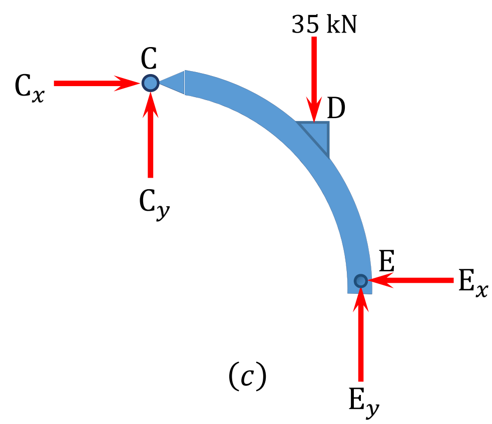


Fig. 6.3. Three – hinged arch.



Solution

The free-body diagrams of the entire arch and its segment *CE* are shown in Figure 6.3b and Figure 6.3c, respectively. Applying the equations of static equilibrium suggests the following:

Entire arch.

$$+\circlearrowleft \sum M_A = 0$$
$$E_y(45) - E_x(15) - 50(5) - 35(40) = 0$$
$$E_y(45) - E_x(15) = 1650$$

Arch segment *CE*.

$$+\circlearrowleft \sum M_C = 0$$
$$E_y(25) - E_x(30) - 35(20) = 0$$
$$E_y(25) - E_x(30) = 700$$

Solving equations 6.1 and 6.2 simultaneously yields the following:

$E_y = 40$ kN $\qquad\qquad E_y = 40 \text{ kN} \uparrow$

$E_x = 10\ \text{kN}$ $\qquad\qquad$ $E_x = 10\ \text{kN} \leftarrow$

Entire arch again.

$+\uparrow \sum F_y = 0$

$A_y + 40 - 50 - 35 = 0$

$A_y = 45\ \text{kN}$ $\qquad\qquad$ $A_y = 45\ \text{kN} \uparrow$

$+\rightarrow \sum F_x = 0$

$A_x - 10 = 0$

$A_x = 10\ \text{kN}$ $\qquad\qquad$ $A_x = 10 \rightarrow$

Example 6.2

A parabolic arch with supports at the same level is subjected to the combined loading shown in Figure 6.4a. Determine the support reactions and the normal thrust and radial shear at a point just to the left of the 150 kN concentrated load.

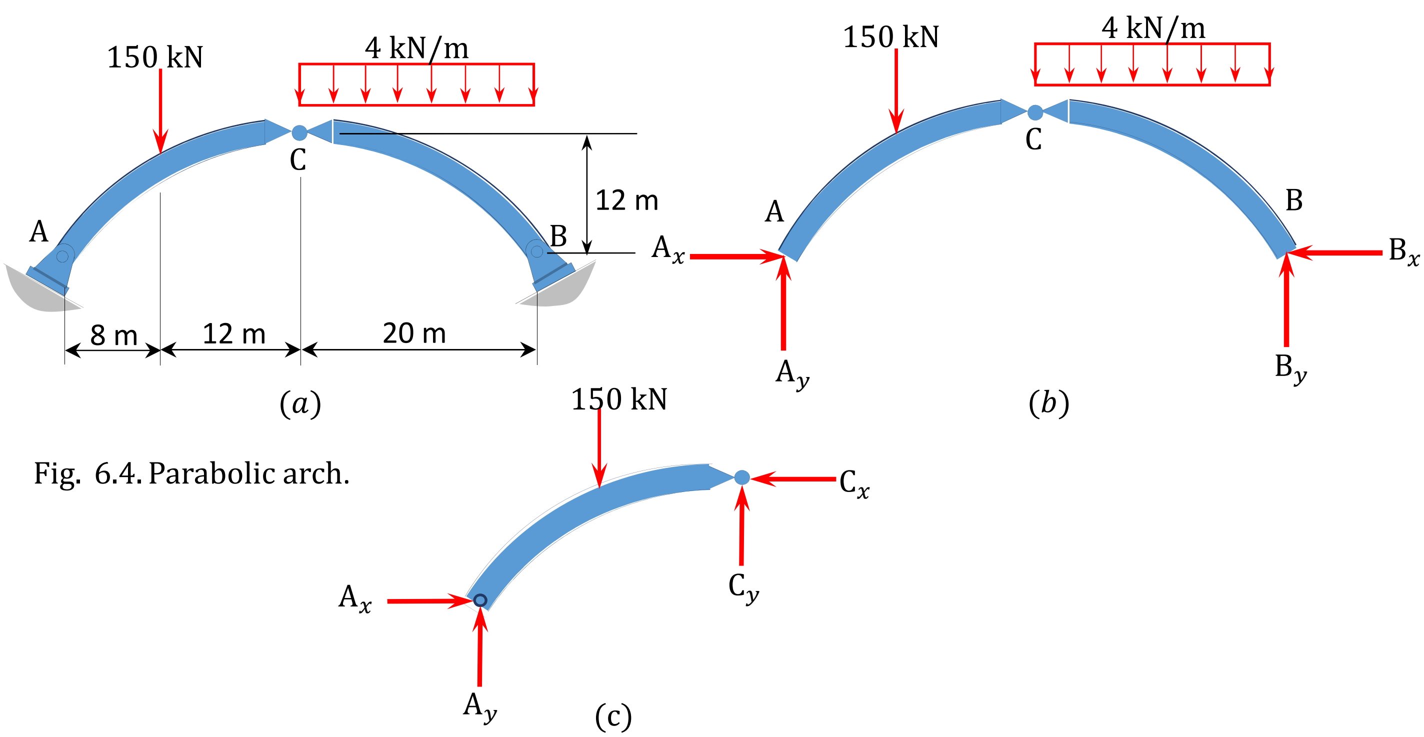


Fig. 6.4. Parabolic arch.

Solution

Support reactions. The free-body diagram of the entire arch is shown in Figure 6.4b, while that of its segment AC is shown in Figure 6.4c. Applying the equations of static equilibrium to determine the arch's support reactions suggests the following:

Entire arch.

$$+\circlearrowleft \sum M_A = 0$$
$$B_y(40) - 150(8) - 4(20)(30) = 0$$
$$B_y = 90 \text{ kN} \qquad B_y = 90 \text{ kN} \uparrow$$
$$+\uparrow \sum F_y = 0$$
$$A_y + 90 - 150 - 4(20) = 0$$
$$A_y = 140 \text{ kN} \qquad A_y = 140 \text{ kN} \uparrow$$

Arch segment AC.

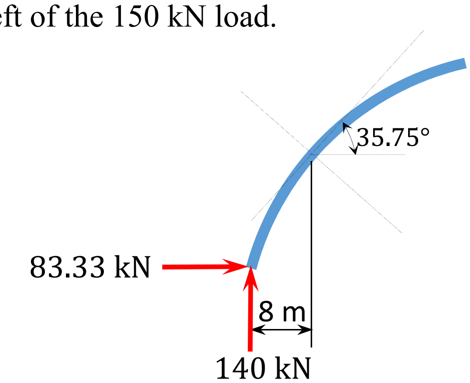
$$+\circlearrowleft \sum M_C = 0$$
$$A_x(12) - 140(20) + 150(12) = 0$$
$$A_x = 83.33 \text{ kN} \qquad A_x = 83.33 \text{ kN} \rightarrow$$

Entire arch again.

$$+\rightarrow \sum F_x = 0$$
$$83.33 - B_x = 0$$
$$B_x = 83.33 \text{ kN} \qquad B_x = 83.33 \text{ kN} \leftarrow$$

Normal thrust and radial shear. To determine the normal thrust and radial shear, find the angle between the horizontal and the arch just to the left of the 150 kN load.

$$y = \frac{4fx}{L^2}(L - x) = \frac{4f}{L^2}(Lx - x^2)$$
$$\tan\theta = y' = \frac{4f}{L^2}(L - 2x)$$
$$= \frac{4(12)}{(40)^2}(40 - 2 \times 8) = 0.72$$
$$= 35.75°$$



(d)

Normal thrust.

$$N = A_y \sin(35.75°) + A_x \cos(35.75°)$$

$$= 140\sin(35.75°) + 83.33\cos(35.75°) = 149.42 \text{ kN} \qquad N = 149.42 \text{ kN}$$

Radial shear.

$$V = A_y\cos(35.75°) + A_x\sin(35.75°)$$
$$= 140\cos(35.75°) - 83.33\sin(35.75°) = 64.93 \text{ kN} \qquad V = 64.93 \text{ kN}$$

Example 6.3

A parabolic arch is subjected to a uniformly distributed load of 600 lb/ft throughout its span, as shown in Figure 6.5a. Determine the support reactions and the bending moment at a section Q in the arch, which is at a distance of 18 ft from the left-hand support.

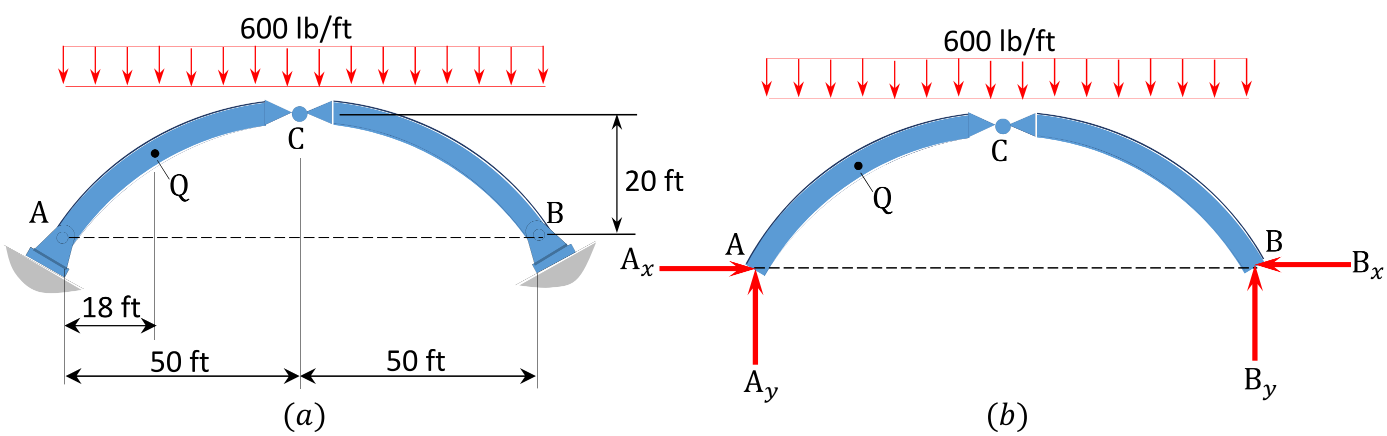
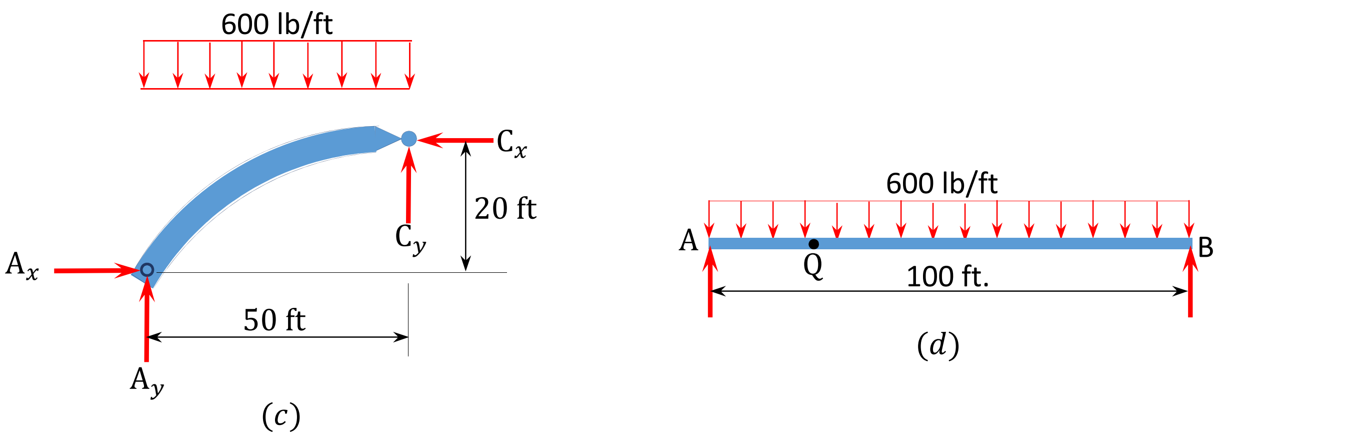


Fig. 6.5. Parabolic arch.



Solution

Support reactions. The free-body diagram of the entire arch is shown in Figure 6.5b, while that of its segment *AC* is shown Figure 6.5c. Applying the equations of static equilibrium for the determination of the arch's support reactions suggests the following:

Free-body diagram of entire arch. Due to symmetry in loading, the vertical reactions in both supports of the arch are the same. Therefore, $A_y = B_y = \frac{wL}{2} = \frac{0.6(100)}{2} = 30 \text{ kips}$

The horizontal thrust at both supports of the arch are the same, and they can be computed by considering the free body diagram in Figure 6.5b. Taking the moment about point *C* of the free-body diagram suggests the following:

Free-body diagram of segment *AC*. The horizontal thrust at both supports of the arch are the same, and they can be computed by considering the free body diagram in Figure 6.5c. Taking the moment about point *C* of the free-body diagram suggests the following:

$$+\curvearrowleft \sum M_C = 0$$

$$A_x(20) - 30(50) + 0.6(50)(25) = 0$$

$$A_x = 37.5 \text{ kips} \qquad A_x = 37.5 \text{ kips} \rightarrow$$

Free-body diagram of entire arch again.

$$+\uparrow \sum F_x = 0$$

$$37.5 - B_x = 0$$

$$B_x = 37.5 \text{ kips} \qquad B_x = 37.5 \text{ kips} \leftarrow$$

Bending moment at point *Q*: To find the bending moment at a point *Q*, which is located 18 ft from support *A*, first determine the ordinate of the arch at that point by using the equation of the ordinate of a parabola.

$$y = \frac{4fx}{L^2}(L - x)$$

$$y_{x=18\text{ft}} = \frac{4(20)(18)}{(100)^2}(100 - 18) = 11.81\text{ft}$$

The moment at *Q* can be determined as the summation of the moment of the forces on the left-hand portion of the point in the beam, as shown in Figure 6.5c, and the moment due to the horizontal thrust, A_x. Thus, $M_Q = A_y(18) - 0.6(18)(9) - A_x(11.81)$

$$= 30(18) - 0.6(18)(9) - 37.5(11.81) = -75 \text{ lb.ft} \qquad M_Q = -75 \text{ lb.ft}$$

A parabolic arch is subjected to two concentrated loads, as shown in Figure 6.6a. Determine the support reactions and draw the bending moment diagram for the arch.

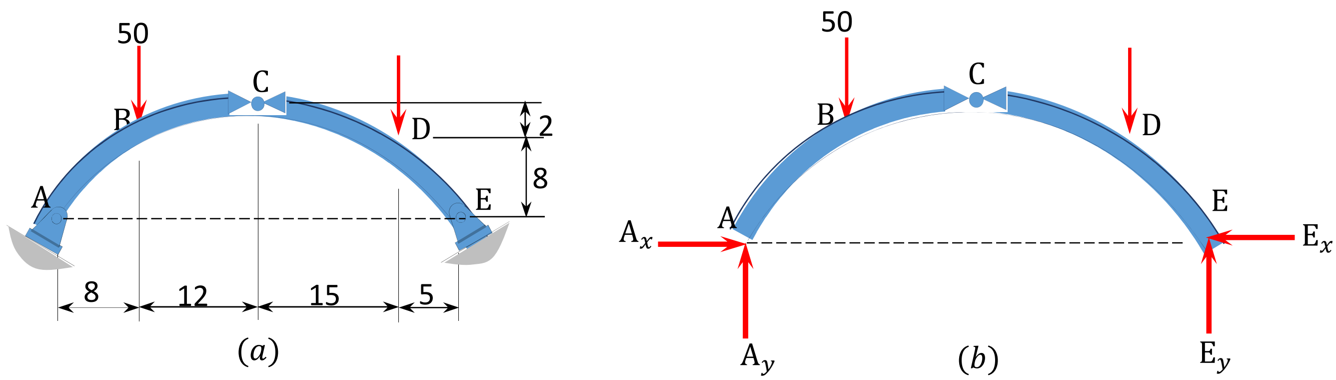
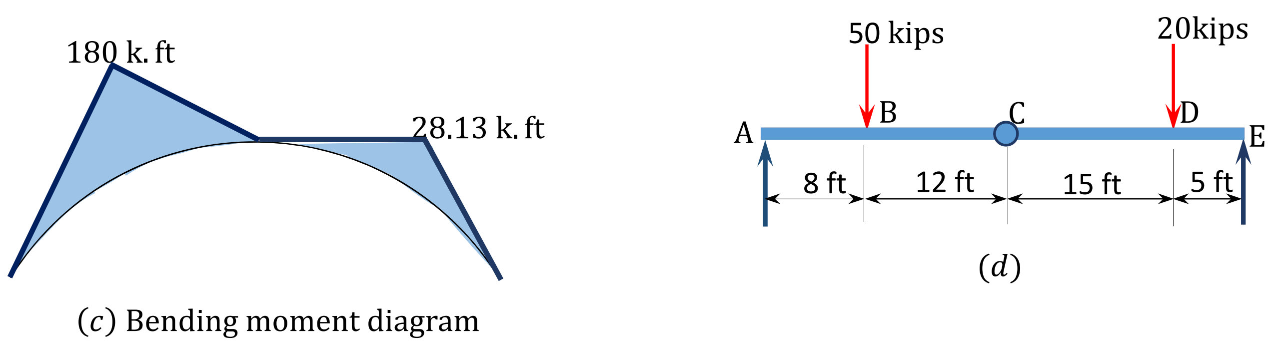


Fig. 6.6. Parabolic arch.



Solution

Support reactions. The free-body diagram of the entire arch is shown in Figure 6.6b. Applying the equations of static equilibrium determines the components of the support reactions and suggests the following:

Entire arch.

$$+\curvearrowleft \sum M_A = 0$$

$$E_y(40) - 50(8) - (20)(35) = 0$$

$E_y = 27.5$ kips $\qquad\qquad E_y = 27.5$ kip $\uparrow$

$$+\uparrow \sum F_y = 0$$

$A_y + 27.5 - 50 - 20 = 0$
$A_y = 42.5 \text{ kips}$ $\qquad A_y = 42.5 \text{ kips} \uparrow$

Arch segment *EC*.

For the horizontal reactions, sum the moments about the hinge at *C*.

$$+\curvearrowleft \sum M_C = 0$$
$$27.5(20) - E_x(10) - 20(15) = 0$$
$$E_x = 25 \text{ kips} \qquad E_x = 25 \text{ kips} \leftarrow$$

Entire arch again.

$$+\uparrow \sum F_x = 0$$
$$-25 + A_x = 0$$
$$A_x = 25 \text{ kips} \qquad A_x = 25 \text{ kips} \rightarrow$$

Bending moment at the locations of concentrated loads. To find the bending moments at sections of the arch subjected to concentrated loads, first determine the ordinates at these sections using the equation of the ordinate of a parabola, which is as follows:

$$y = \frac{4fx}{L^2}(L - x)$$

$$y_{x = 8\text{ft}} = \frac{4(10)(8)}{(40)^2}(40 - 8) = 6.4 \text{ ft}$$

$$y_{x = 5\text{ft}} = \frac{4(10)(5)}{(40)^2}(40 - 5) = 4.375 \text{ ft}$$

When considering the beam in Figure 6.6d, the bending moments at *B* and *D* can be determined as follows:

$$M_B = A_y(8) - A_x(6.4)$$
$$= 42.5(8) - 25(6.4) = 180 \text{ k.ft} \qquad M_B = 180 \text{ k.ft}$$

$$M_D = E_y(5) - E_x(4.375)$$
$$= 27.5(5) - 25(4.375) = 28.13 \text{ k.ft} \qquad M_D = 28.13 \text{ k.ft}$$

6.2 Cables

Cables are flexible structures that support the applied transverse loads by the tensile resistance developed in its members. Cables are used in suspension bridges, tension leg offshore platforms, transmission lines, and several other engineering applications. The distinguishing feature of a cable is its ability to take different shapes when subjected to different types of loadings. Under a uniform load, a cable takes the shape of a curve, while under a concentrated load, it takes the form of several linear segments between the load's points of application.

6.2.1 General Cable Theorem

The general cable theorem states that at any point on a cable that is supported at two ends and subjected to vertical transverse loads, the product of the horizontal component of the cable tension and the vertical distance from that point to the cable chord equals the moment which would occur at that section if the load carried by the cable were acting on a simply supported beam of the same span as that of the cable.

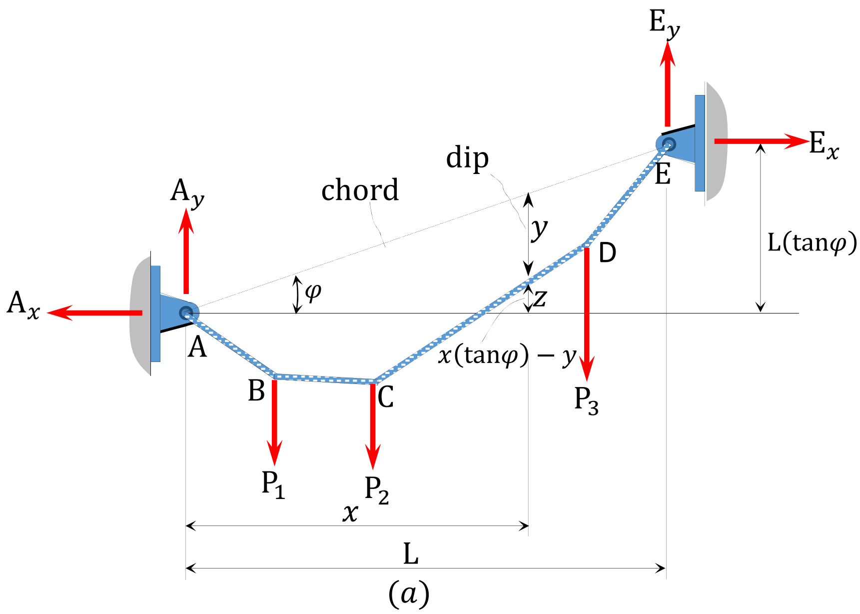
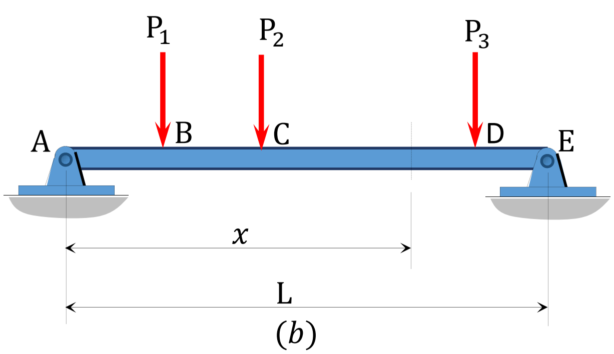


Fig. 6.7. Cable (a) and beam (b).



To prove the general cable theorem, consider the cable and the beam shown in Figure 6.7a and Figure 6.7b, respectively. Both structures are supported at both ends, have a span L, and are subjected to the same concentrated loads at B, C, and D. A line joining supports A and E is referred to as the chord, while a vertical height from the chord to the surface of the cable at any point of a distance x from the left support, as shown in Figure 6.7a, is known as the dip at that point. For equilibrium of a structure, the horizontal reactions at both supports must be the same. From static equilibrium, the moment of the forces on the cable about support B and about the section at a distance x from the left support can be expressed as follows, respectively:

$+\curvearrowleft \sum M_B = 0$

$$-A_y L - A_x L(\tan\varphi) + \sum M_{BP} = 0 \tag{6.6}$$

where

$\sum M_{BP}$ = the algebraic sum of the moment of the applied forces about support B.

$+\curvearrowleft \sum M_x = 0$

$$-A_y x - A_x[x\tan\varphi - y] + \sum M_{xP} = 0 \tag{6.7}$$

Solving equation 6.1 suggest that $A_y = \frac{[\sum M_{BP} - A_x L tan\varphi]}{L}$ (6.8)

Substituting A_y from equation 6.8 into equation 6.7 suggests the following: $\frac{[\sum M_{BP} - A_x L tan\varphi]x}{L} + A_x(x\tan\varphi - y) = \sum M_{xP}$

or $\frac{x\sum M_{BP}}{L} - xA_x\tan\varphi + xA_x\tan\varphi - A_x y = \sum M_{xP}$

or

$$A_x y = \frac{x\sum M_{BP}}{L} - \sum M_{xP} \tag{6.9}$$

To obtain the expression for the moment at a section x from the right support, consider the beam in Figure 6.7b. First, determine the reaction at A using the equation of static equilibrium as follows:

$$\sum M_B = 0$$
$$A_y = \frac{\sum M_{BP}}{L} \tag{6.10}$$

The moment at a section of the beam at a distance x from support $A = A_y x - \sum M_{xP}$ (6.11)

Substituting A_y from equation 6.10 into equation 6.11 suggests the following:

The moment at section $x = \frac{x\sum M_{BP}}{L} - \sum M_{xP}$ (6.12)

The moment at a section of a beam at a distance x from the left support presented in equation 6.12 is the same as equation 6.9. This confirms the general cable theorem.

Example 6.5

A cable supports two concentrated loads at B and C, as shown in Figure 6.8a. Determine the sag at B, the tension in the cable, and the length of the cable.

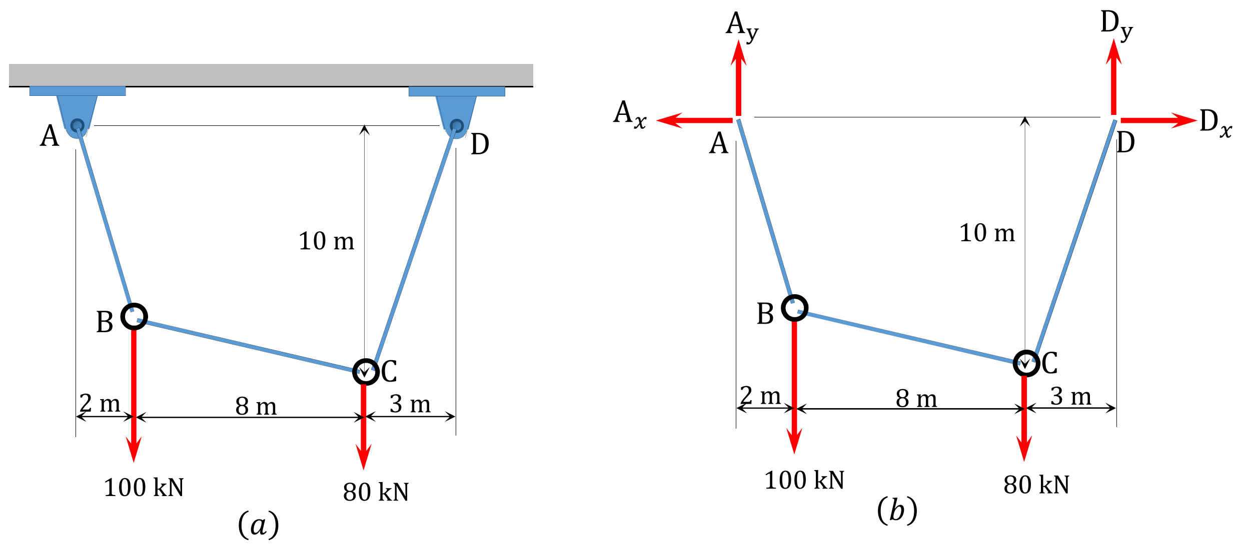


Fig. 6.8. Cable.

Solution

Support reactions. The reactions of the cable are determined by applying the equations of equilibrium to the free-body diagram of the cable shown in Figure 6.8b, which is written as follows:

$$+\curvearrowleft \sum M_A = 0$$

$$-100(2) - 80(10) + 13D_y = 0$$

$$D_y = 76.92 \text{ kN}$$

$$+\uparrow \sum F_y = 0$$

$$A_y + 76.92 - 100 - 80 = 0$$

$$A_y = 103.08 \text{ kN}$$

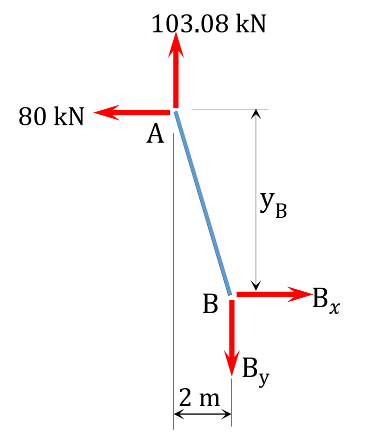
$$+\curvearrowleft \sum M_C = 0$$

$$-A_x(10) + 100(8) = 0$$

$$A_x = 80 \text{ kN}$$

$$+\rightarrow \sum F_x = 0$$
$$-D_x + 80 = 0$$
$$D_x = 80 \text{ kN}$$

Sag at *B*. The sag at point *B* of the cable is determined by taking the moment about *B*, as shown in the free-body diagram in Figure 6.8c, which is written as follows:



(c)

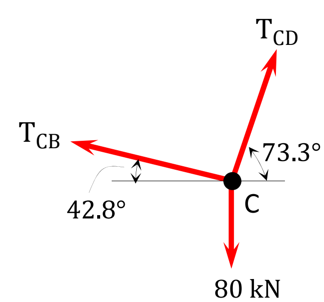
$$+\circlearrowleft \sum M_B = 0$$
$$-A_y(2) + A_x(y_B) = 0$$
$$y_B = \frac{A_y(2)}{A_x} = \frac{103.08(2)}{80} = 2.58 \text{ m} \qquad y_B = 2.58 \text{ m}$$

Tension in cable.

Tension at *A* and *D*.

$$T_A = T_{AB} = \sqrt{(A_y)^2 + (A_x)^2} = \sqrt{(103.08)^2 + (80)^2} = 130.48 \text{ kN}$$

$$T_D = T_{DC} = \sqrt{(D_y)^2 + (D_x)^2} = \sqrt{(76.92)^2 + (80)^2} = 110.98 \text{ kN}$$



(d)

Tension in segment *CB*.

$$+\rightarrow \sum F_x = 0$$
$$T_{CD}\cos 73.3° - T_{CB}\cos 42.8° = 0$$
$$T_{CB} = \frac{T_{CD}\cos(73.3°)}{\cos 42.8} = \frac{110.98\cos(73.3°)}{\cos 42.8} = 43.46 \text{ kN}$$

Length of cable. The length of the cable is determined as the algebraic sum of the lengths of the segments. The lengths of the segments can be obtained by the application of the Pythagoras theorem, as follows:

$$L = \sqrt{(2.58)^2 + (2)^2} + \sqrt{(10 - 2.58)^2 + (8)^2} + \sqrt{(10)^2 + (3)^2} = 24.62 \text{ m}$$

Example 6.6

A cable supports three concentrated loads at *B*, *C*, and *D*, as shown in Figure 6.9a. Determine the sag at *B* and *D*, as well as the tension in each segment of the cable.

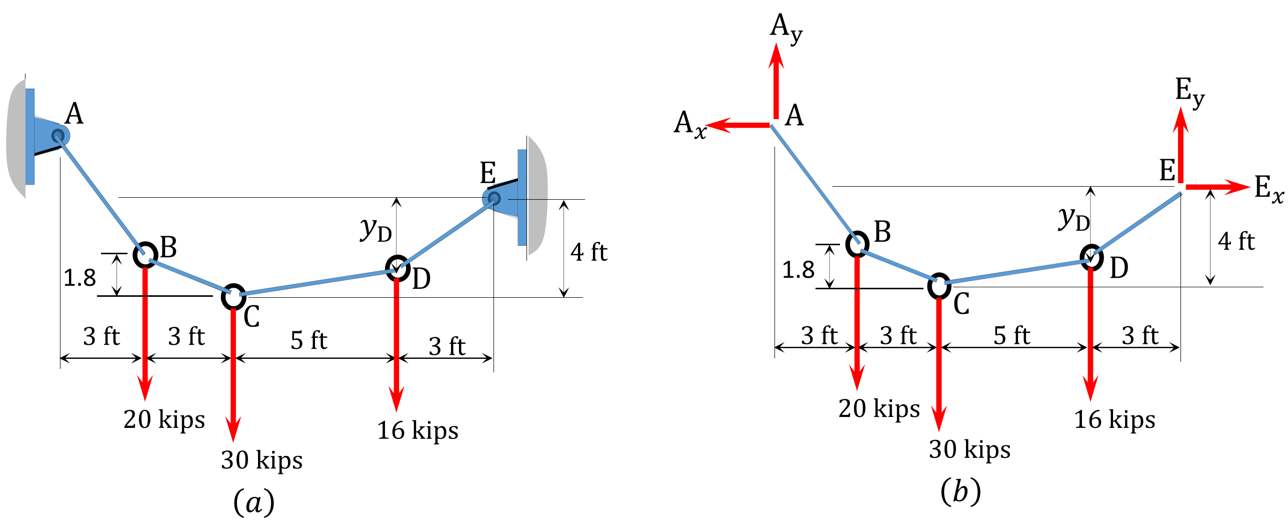


Fig. 6.9. Cable.

Solution

Support reactions. The reactions shown in the free-body diagram of the cable in Figure 6.9b are determined by applying the equations of equilibrium, which are written as follows:

$$+\circlearrowleft \sum M_A = 0$$

$$-20(3) - 30(6) - 16(11) + 14 = 0$$

$$E_y = 29.71 \text{ kips}$$

$$+\uparrow \sum F_y = 0$$

$$A_y + 29.71 - 20 - 30 - 16 = 0$$

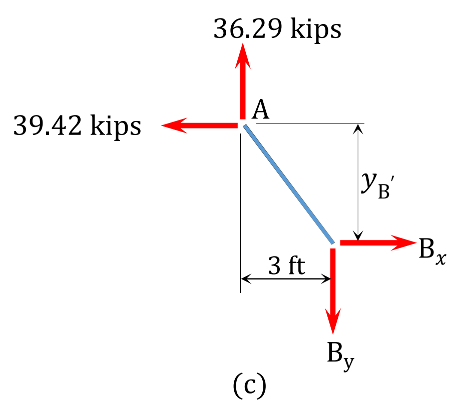
$$A_y = 36.29 \text{ kips}$$

$$+\circlearrowleft \sum M_C = 0$$

$$29.71(8) - E_x(4) - 16(5) = 0$$

$$E_x = 39.42 \text{ kips}$$

$$+\rightarrow \sum F_x = 0$$

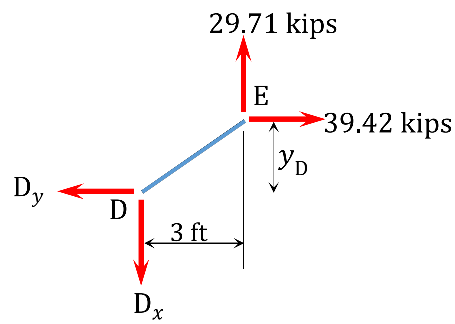


$$-A_x + 39.42 = 0$$
$$A_x = 39.42 \text{ kips}$$

Sag. The sag at *B* is determined by summing the moment about *B*, as shown in the free-body diagram in Figure 6.9c, while the sag at *D* was computed by summing the moment about *D*, as shown in the free-body diagram in Figure 6.9d.

Sag at *B*.

$$\curvearrowleft + \sum M_B = 0$$
$$-36.29(3) + 39.42(y_{B'}) = 0$$
$$y_{B'} = 2.76 \text{ ft.}$$



(d)

Sag at *D*.

$$\curvearrowleft + \sum M_D = 0$$
$$29.71(3) + 39.42(y_D) = 0$$
$$y_D = 2.26 \text{ ft.}$$

Tension.

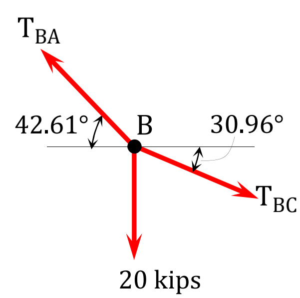
Tension at *A*.

$$T_A = T_{AB} = \sqrt{(A_y)^2 + (A_x)^2} = \sqrt{(36.29)^2 + (39.42)^2} = 53.58 \text{ kips}$$

Tension at *E*.

$$T_E = T_{ED} = \sqrt{(E_y)^2 + (E_x)^2} = \sqrt{(29.71)^2 + (39.42)^2} = 49.36 \text{ kips}$$

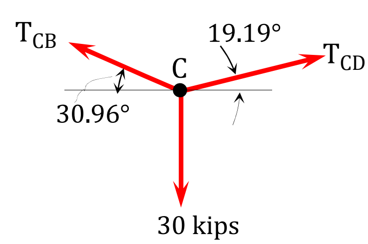
Tension at *B*.



(e)

$$\rightarrow + \sum F_x = 0$$
$$-T_{BA}\cos 42.61° + T_{BC}\cos 30.96° = 0$$
$$T_{BC} = \frac{T_{BA}\cos 42.61°}{\cos 30.96°} = \frac{53.58\cos 42.61°}{\cos 30.96°} = 46 \text{ kips}$$

Tension at *C*.



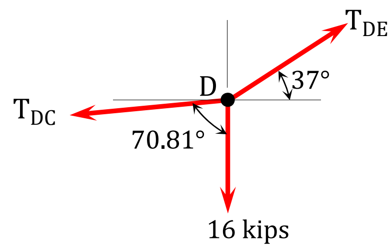
(f)

$$\rightarrow + \sum F_x = 0$$

$-T_{CB}\cos 30.96° + T_{CD}\cos 19.19° = 0$

$T_{CD} = \frac{T_{CB}\cos 30.96°}{\cos 19.19°} = \frac{46\cos 30.96°}{\cos 19.19°} = 41.77 \text{ kips}$

Tension at *D*.



(g)

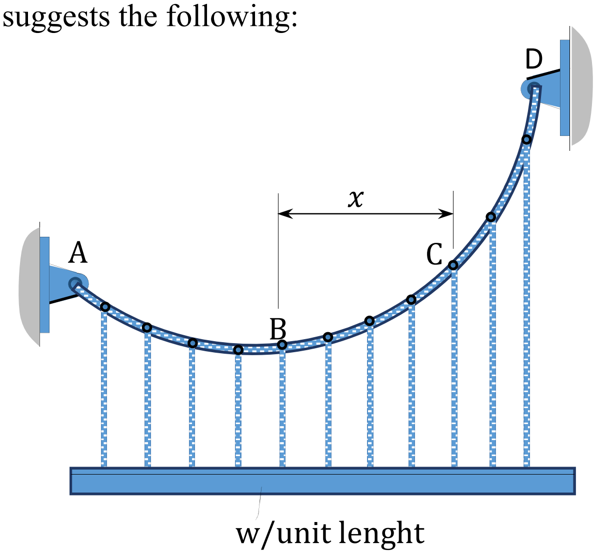
$\rightarrow + \sum F_x = 0$

$-T_{DC}\sin 70.81° + T_{DE}\cos 37° = 0$

$T_{DE} = \frac{T_{DC}\sin(70.81°)}{\cos 37°} = \frac{41.77\sin(70.81°)}{\cos 37°} = 49.40 \text{ kN}$

6.2.2 Parabolic Cable Carrying Horizontal Distributed Loads

To develop the basic relationships for the analysis of parabolic cables, consider segment *BC* of the cable suspended from two points *A* and *D*, as shown in Figure 6.10a. Point *B* is the lowest point of the cable, while point *C* is an arbitrary point lying on the cable. Taking *B* as the origin and denoting the tensile horizontal force at this origin as T_0 and denoting the tensile inclined force at *C* as T, as shown in Figure 6.10b, suggests the following:



(a)

Fig. 6.10. Suspended cable.

Figure 6.10c suggests the following:

$$\tan\theta = \frac{dy}{dx} = \frac{wx}{T_0} \quad (6.13)$$

Equation 6.13 defines the slope of the curve of the cable with respect to *x*. To determine the vertical distance between the lowest point of the cable (point *B*) and the arbitrary point *C*, rearrange and further integrate equation 6.13, as follows:

$$dy = \frac{wx}{T_0}dx$$

$$\int_0^y dy = \int_0^x \frac{wx}{T_0}dx$$

$$y = \frac{wx^2}{2T_o} \qquad (6.14)$$

Summing the moments about C in Figure 6.10b suggests the following:

$\curvearrowleft + \sum M_c$

$$wx\left(\frac{x}{2}\right) - T_0 y = 0$$

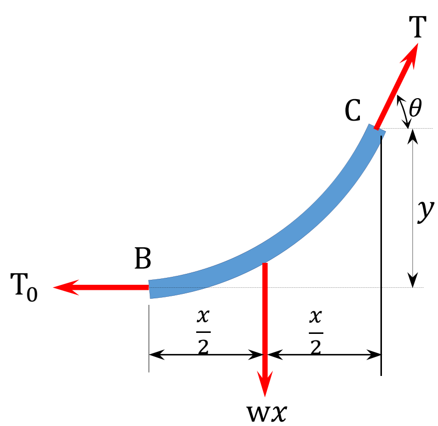
Therefore, $y = \frac{wx^2}{2T_0}$

Applying Pythagorean theory to Figure 6.10c suggests the following:

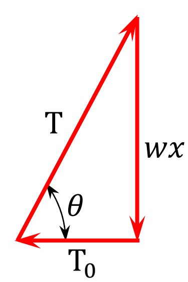
$$T = \sqrt{(T_0)^2 + (wx)^2} \qquad (6.15)$$

where

T and T_0 are the maximum and minimum tensions in the cable, respectively.



(b)



(c)

Example 6.7

A cable supports a uniformly distributed load, as shown Figure 6.11a. Determine the horizontal reaction at the supports of the cable, the expression of the shape of the cable, and the length of the cable.

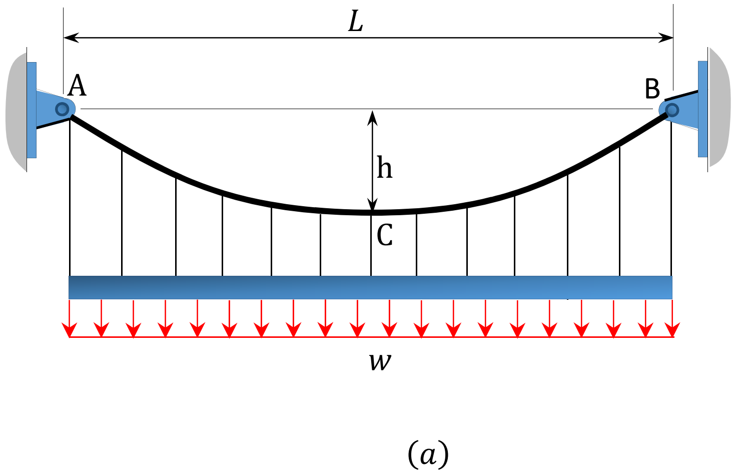
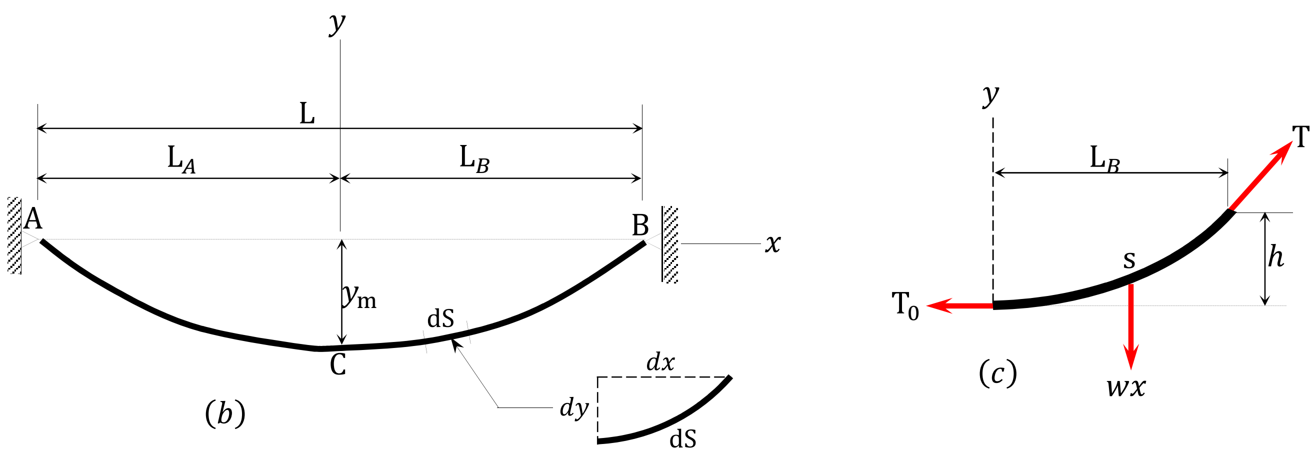


Fig. 6.11. Cable with uniformly distributed load.



Solution

As the dip of the cable is known, apply the general cable theorem to find the horizontal reaction.

At point C, $x = \frac{L}{2}$, $y = h$

The expression of the shape of the cable is found using the following equations:

$$\sum M_{xP} = \left(\frac{wL}{2}\right)\left(\frac{L}{4}\right) = \frac{wL^2}{8}$$

$$\sum M_{BP} = \frac{wL^2}{2}$$

$$A_x h = \left(\frac{1/2L}{L}\right)\left(\frac{wL^2}{2}\right) - \left(\frac{wL^2}{8}\right)$$

$$A_x = \frac{wL^2}{8h}$$

For any point $P(x, y)$ on the cable, apply cable equation.

The moment at any section *x* due to the applied load is expressed as follows:

$$\sum M_x = \frac{wx^2}{2}$$

The moment at support *B* is written as follows:

$$\sum M_B = \frac{wL^2}{2}$$

Applying the general cable theorem yields the following:

$$A_x y = \left(\frac{x}{L}\right)\left(\frac{wL^2}{2}\right) - \frac{wx^2}{2}$$
$$= \left(\frac{w}{2}\right)(x)(L - x)$$
$$\left(\frac{wL^2}{8h}\right)y = \left(\frac{w}{2}\right)(x)(L - x)$$

$$y = \frac{4h}{L^2}x(L - x)$$

The length of the cable can be found using the following:

$$(dS)^2 = (dx)^2 + (dy)^2$$

$$(dS)^2 = (dx)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$
$$S = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_0^{L_B} dS = \int_0^{L_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{L_B} \sqrt{1 + \left(\frac{wx}{T_0}\right)^2}\, dx \qquad (6.16)$$

The solution of equation 6.16 can be simplified by expressing the radical under the integral as a series using a binomial expansion, as presented in equation 6.17, and then integrating each term.

$$\sqrt{1 + a} = (1 + a)^{1/2} = 1 + \frac{1}{2}a - \frac{1}{8}a^2 + \frac{1}{16}a^3 - + \frac{5}{128}a^4 + \frac{7}{256}a^5 - \cdots \qquad (6.17)$$

Putting $a = \left(\frac{wx}{T_0}\right)^2$ into three terms of the expansion in equation 6.13 suggests the following:

$$\sqrt{1 + \left(\frac{wx}{T_0}\right)^2} = 1 + \frac{1}{2}\left(\frac{w}{T_0}\right)^2 X^2 - \frac{1}{8}\left(\frac{w}{T_0}\right)^4 X^4 \qquad (6.18)$$

Thus, equation 6.16 can be written as the following:

$$S = \int_0^{L_B} dS = \int_0^{L_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{L_B} \left[1 + \frac{1}{2}\left(\frac{w}{T_0}\right)^2 X^2 - \frac{1}{8}\left(\frac{w}{T_0}\right)^4 X^4\right] dx$$

$$= \left[X + \frac{1}{6}\left(\frac{w}{T_0}\right)^2 X^3 - \frac{1}{40}\left(\frac{w}{T_0}\right)^4 X^5 \right] \Bigg|_0^{L_B}$$

$$= L_B + \frac{1}{6}\left(\frac{w}{T_0}\right)^2 L_B^3 - \frac{1}{40}\left(\frac{w}{T_0}\right)^4 L_B^5 \tag{6.19}$$

Putting $T_0 = \frac{wL_B^2}{2h}$ into equation 6.19 suggests:

$$S = L_B + \frac{4h^2}{6L_B} - \frac{16h^4}{40L_B^3}$$

$$= L_B \left[1 + \frac{2}{3}\left(\frac{h}{L_B}\right)^2 - \frac{2}{5}\left(\frac{h}{L_B}\right)^4 \right] \tag{6.20}$$

Example 6.8

A cable subjected to a uniform load of 240 N/m is suspended between two supports at the same level 20 m apart, as shown in Figure 6.12. If the cable has a central sag of 4 m, determine the horizontal reactions at the supports, the minimum and maximum tension in the cable, and the total length of the cable.

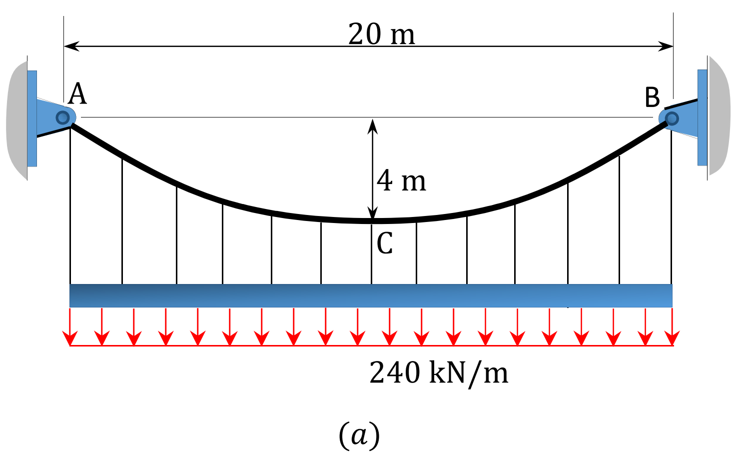
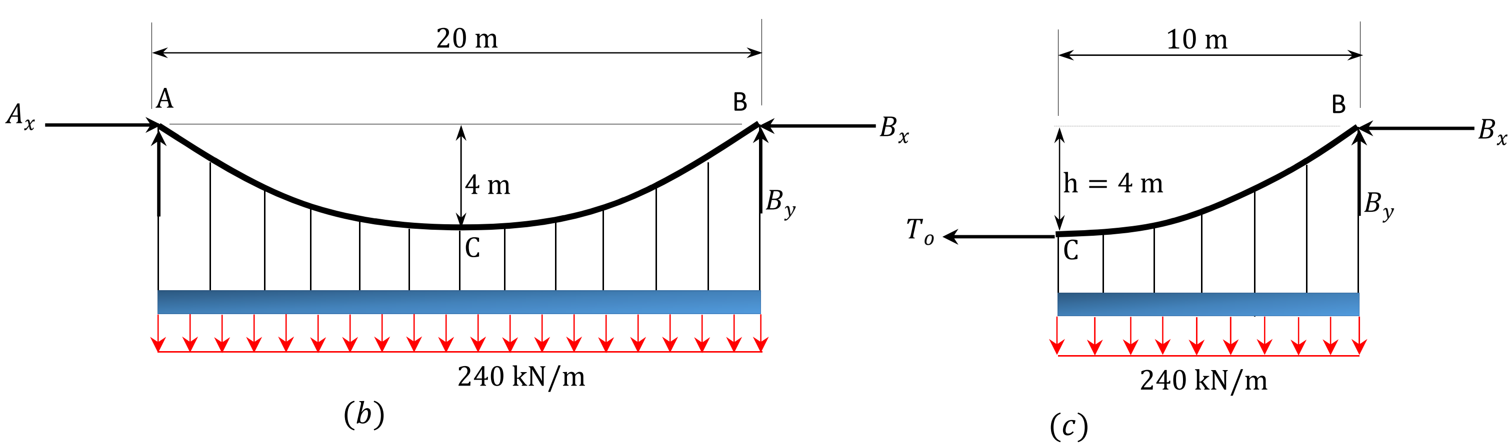


Fig. 6.12. Cable.



Solution

Horizontal reactions. Applying the general cable theorem at point *C* suggests the following:

when $x = \frac{L}{2}, h = 4$ m

$$+\curvearrowleft \sum M_x = \frac{wL^2}{8} = \frac{240(20)^2}{8} = 12000 \text{ Nm}$$

$$+\curvearrowleft \sum M_B = \frac{wL^2}{2} = \frac{240(20)^2}{2} = 48000 \text{ Nm}$$

$$A_x(4) = 48000 - 12000$$

$$A_x = B_x = 9000 \text{ N}$$

Minimum and maximum tension. From the free-body diagram in Figure 6.12c, the minimum tension is as follows:

$$\curvearrowleft + \sum M_c$$

$$wx\left(\frac{x}{2}\right) - T_0 h = 0$$

Therefore, $T_0 = \frac{wx^2}{2h} = \frac{240(10)^2}{2(4)} = 3000\ N$

From equation 6.15, the maximum tension is found, as follows:

$$T_{max} = \sqrt{(T_0)^2 + (wx)^2} = \sqrt{(3000)^2 + (240 \times 10)^2} = 3841.87 \text{ N}$$

The total length of cable:

$$S = (2)(10)\left[1 + \tfrac{2}{3}\left(\tfrac{4}{10}\right)^2 - \tfrac{2}{5}\left(\tfrac{4}{10}\right)^4\right]$$

$$= (20)\left[1 + \tfrac{2}{3}\left(\tfrac{4}{10}\right)^2 - \tfrac{2}{5}\left(\tfrac{4}{10}\right)^4\right]$$

$$= 21.93 \text{ m}$$

Chapter Summary

Internal forces in arches and cables: Arches are aesthetically pleasant structures consisting of curvilinear members. They are used for large-span structures. The presence of horizontal thrusts at the supports of arches results in the reduction of internal forces in it members. The lesser shear forces and bending moments at any section of the arches results in smaller member sizes and a more economical design compared with beam design.

Arches: Arches can be classified as two-pinned arches, three-pinned arches, or fixed arches based on their support and connection of members, as well as parabolic, segmental, or circular based on their shapes. Arches can also be classified as determinate or indeterminate. Three-pinned arches are determinate, while two-pinned arches and fixed arches, as shown in Figure 6.1, are indeterminate structures.

Cables: Cables are flexible structures in pure tension. They are used in different engineering applications, such as bridges and offshore platforms. They take different shapes, depending on the type of loading. Under concentrated loads, they take the form of segments between the loads, while under uniform loads, they take the shape of a curve, as shown below.

Some numerical examples have been solved in this chapter to demonstrate the procedures and theorem for the analysis of arches and cables.

Practice Problems

6.1 Determine the reactions at supports *B* and *E* of the three-hinged circular arch shown in Figure P6.1.

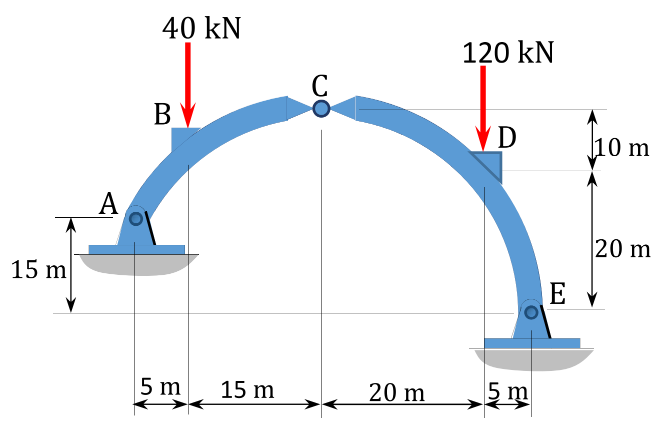


Fig. P6.1. Three − hinged circular arch.

6.2 Determine the reactions at supports *A* and *B* of the parabolic arch shown in Figure P6.2. Also draw the bending moment diagram for the arch.

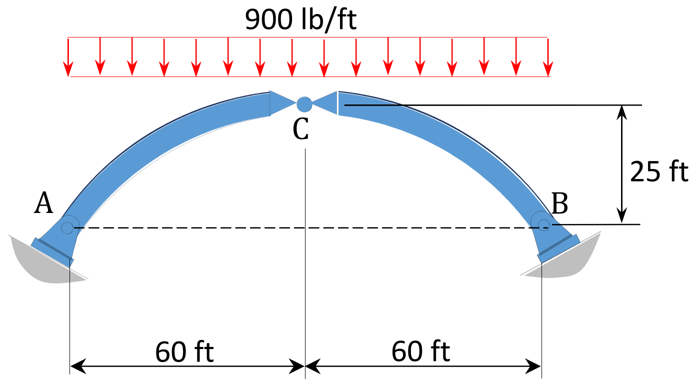
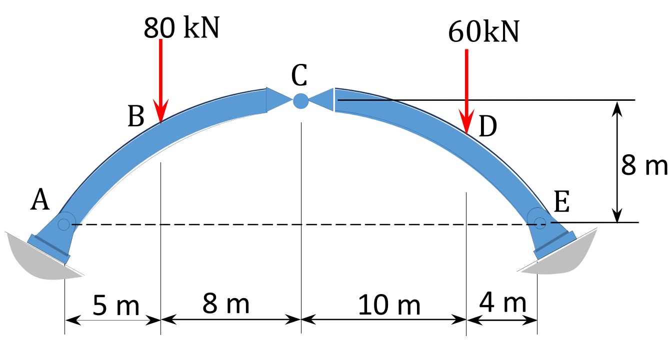


Fig. P6.2. Parabolic arch.

6.3 Determine the shear force, axial force, and bending moment at a point under the 80 kN load on the parabolic arch shown in Figure P6.3.



(a)

Fig. P6.3. Parabolic arch.

6.4 In Figure P6.4, a cable supports loads at point *B* and *C*. Determine the sag at point *C* and the maximum tension in the cable.

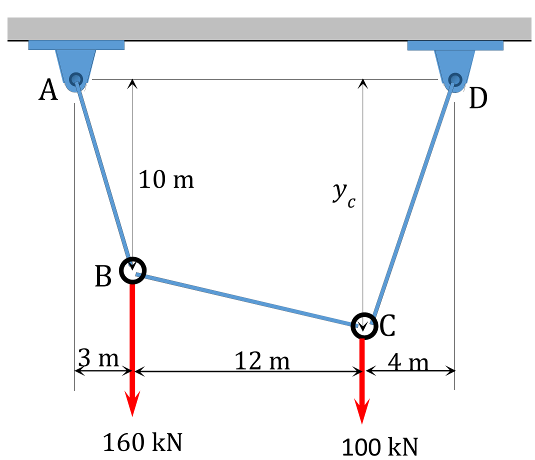


Fig. P6.4. Cable.

6.5 A cable supports three concentrated loads at points *B*, *C*, and *D* in Figure P6.5. Determine the total length of the cable and the length of each segment.

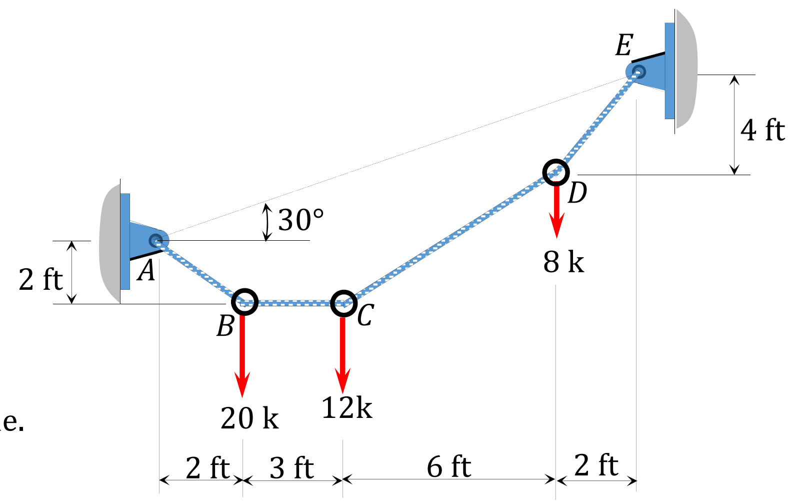


Fig. P6.5. Cable.

6.6 A cable is subjected to the loading shown in Figure P6.6. Determine the total length of the cable and the tension at each support.

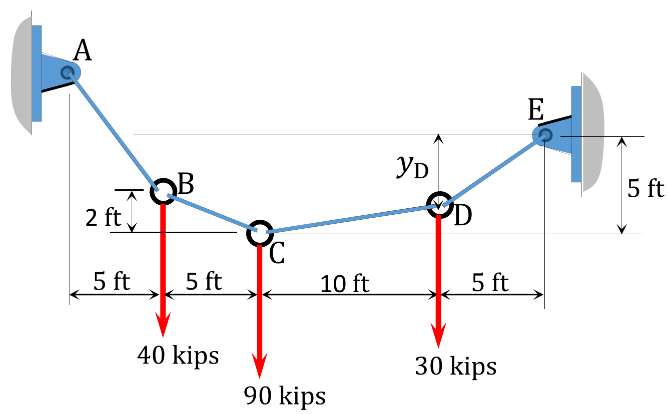


Fig. P6.6. Cable.

6.7 A cable shown in Figure P6.7 supports a uniformly distributed load of 100 kN/m. Determine the tensions at supports *A* and *C* at the lowest point *B*.

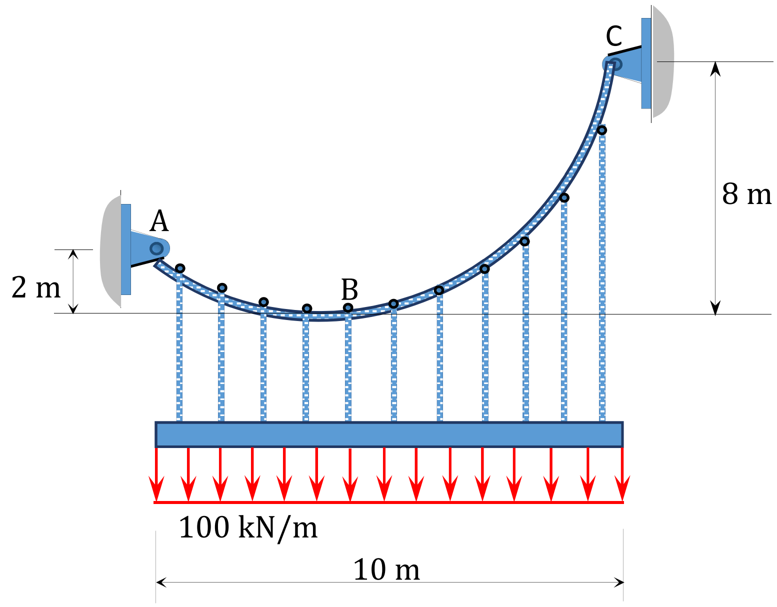


Fig. P6.7. Cable.

6.8 A cable supports a uniformly distributed load in Figure P6.8. Find the horizontal reaction at the supports of the cable, the equation of the shape of the cable, the minimum and maximum tension in the cable, and the length of the cable.

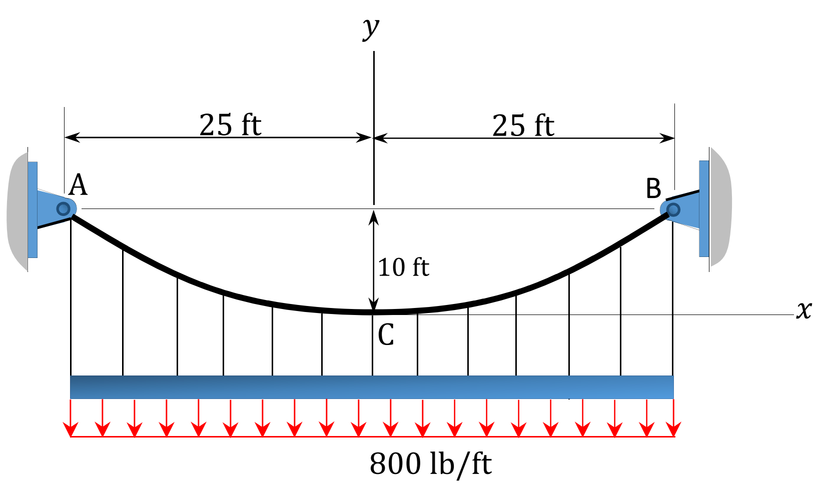


Fig. P6.8. Cable.

6.9 A cable subjected to a uniform load of 300 N/m is suspended between two supports at the same level 20 m apart, as shown in Figure P6.9. If the cable has a central sag of 3 m, determine the horizontal reactions at the supports, the minimum and maximum tension in the cable, and the total length of the cable.

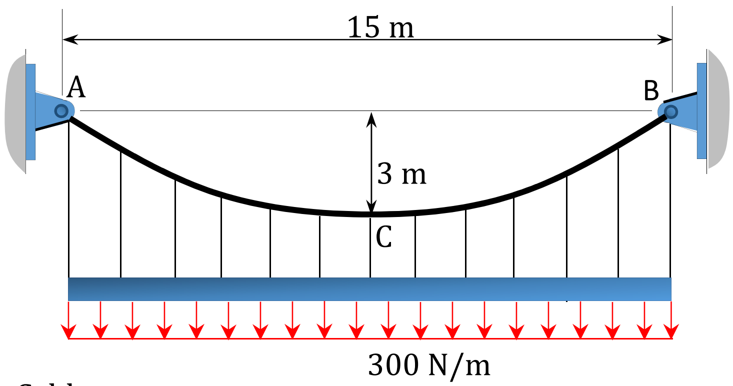


Fig. P6.9. Cable.

Chapter 7

Deflection of Beams: Geometric Methods

7.1 Introduction

The serviceability requirements limit the maximum deflection that is allowed in a structural element subjected to external loading. Excessive deflection may result in the discomfort of the occupancy of a given structure and can also mar its aesthetics. Most codes and standards provide the maximum allowable deflection for dead loads and superimposed live loads. To ensure that the possible maximum deflection that could occur under a given loading is within acceptable value, the structural component is usually analyzed for deflection, and the determined maximum deflection value is compared with the specified values in the codes and standards of practice.
There are several methods of determining the deflection of a beam or frame. The choice of a particular method is dependent on the nature of the loading and the type of problem being solved. Some of the methods used in this chapter include the method of double integration, the method of singularity function, the moment-area method, the unit-load method, the virtual work method, and the energy methods.

7.2 Derivation of the Equation of the Elastic Curve of a Beam

The elastic curve of a beam is the axis of a deflected beam, as indicated in Figure 7.1a.

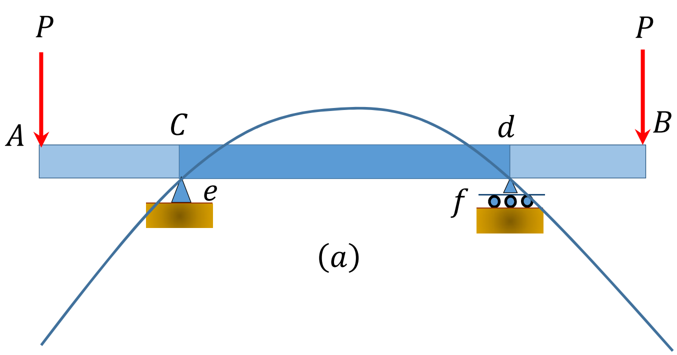
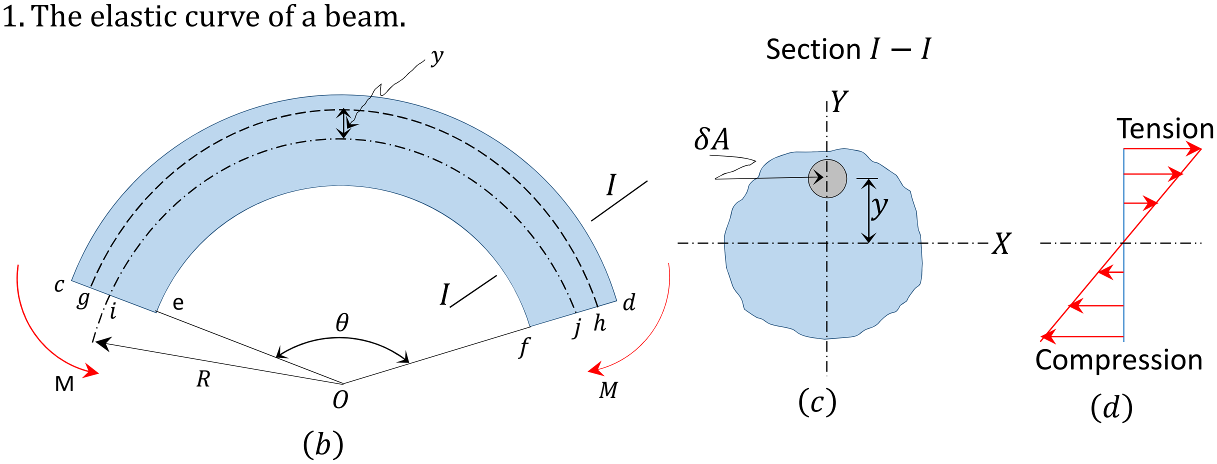


Fig. 7.1. The elastic curve of a beam.



To derive the equation of the elastic curve of a beam, first derive the equation of bending.

Consider the portion $cdef$ of the beam shown in Figure 7.1a, subjected to pure moment, M, for the derivation of the equation of bending. Due to the applied moment M, the fibers above the neutral axis of the beam will elongate, while those below the neutral axis will shorten. Let O be the center and R be the radius of the beam's curvature, and let ij be the axis of the curved beam. The beam subtends an angle θ at O. And let σ be the longitudinal stress in a filament gh at a distance y from the neutral axis.

From geometry, the length of the neutral axis of the beam ij and that of the filament gh, located at a distance y from the neutral axis of the beam, can be computed as follows:

$$ij = R\theta \text{ and } gh = (R + y)\theta$$

The strain ε in the filament can be computed as follows:

$$\varepsilon = \frac{gh - ij}{ij} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y\theta}{R\theta} = \frac{y}{R} \tag{7.1}$$

For a linear elastic material, in which Hooke's law applies, equation 7.1 can be written as follows:

$$\frac{\sigma}{E} = \frac{y}{R} \tag{7.2}$$

If an elementary area δA at a distance y from the neutral axis of the beam (see Figure 7.1c) is subjected to a bending stress σ, the elemental force on this area can be computed as follows:

$$\delta P = \sigma \delta A \tag{7.3}$$

The force on the entire cross-section of the beam then becomes:

$$P = \int \sigma \delta A \tag{7.4}$$

From static equilibrium consideration, the external moment M in the beam is balanced by the moments about the neutral axis of the internal forces developed at a section of the beam. Thus,

$$M = \int (\sigma \delta A) y \tag{7.5}$$

Substituting $\sigma = \frac{Ey}{R}$ from equation 7.2 into equation 7.5 suggests the following:

$$\begin{aligned} M &= \int \left(\frac{E}{R}\right)(y)(y)(\delta A) \\ &= \left(\frac{E}{R}\right) \int y^2 \delta A \end{aligned} \tag{7.6}$$

Putting $I = \int y^2 \delta A$ into equation 7.6 suggests the following:

$$M = \frac{EI}{R} \tag{7.7}$$

where

I = the moment of inertia or the second moment of area of the section.

Combining equations 7.2 and 7.7 suggests the following:

$$\frac{M}{I} = \frac{E}{R} \tag{7.8}$$

The equation of the elastic curve of a beam can be found using the following methods.

From differential calculus, the curvature at any point along a curve can be expressed as follows:

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \tag{7.9}$$

where

$\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are the first and second derivative of the function representing the curve in terms of the Cartesian coordinates x and y.

Since the beam in Figure 7.1 is assumed to be homogeneous and behaves in a linear elastic manner, its deflection under bending is small. Therefore, the quantity $\frac{dy}{dx}$, which represents the slope of the curve at any point of the deformed beam, will also be small. Since $\left(\frac{dy}{dx}\right)^2$ is negligibly insignificant, equation 7.9 could be simplified as follows:

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{[1+0]^{3/2}} = \frac{d^2y}{dx^2} \tag{7.10}$$

Combining equations 7.2 and 7.10 suggests the following:

$$\frac{1}{R} = \frac{M}{EI} = \frac{d^2y}{dx^2} \tag{7.11}$$

Rearranging equation 7.11 yields the following:

$$EI\frac{d^2y}{dx^2} = M \tag{7.12}$$

Equation 7.12 is referred to as the differential equation of the elastic curve of a beam.

7.3 Deflection by Method of Double Integration

Deflection by double integration is also referred to as deflection by the method of direct or constant integration. This method entails obtaining the deflection of a beam by integrating the differential equation of the elastic curve of a beam twice and using boundary conditions to determine the constants of integration. The first integration yields the slope, and the second integration gives the deflection. This method is best when there is a continuity in the applied loading.

Example 7.1

A cantilever beam is subjected to a combination of loading, as shown in Figure 7.2a. Using the method of double integration, determine the slope and the deflection at the free end.

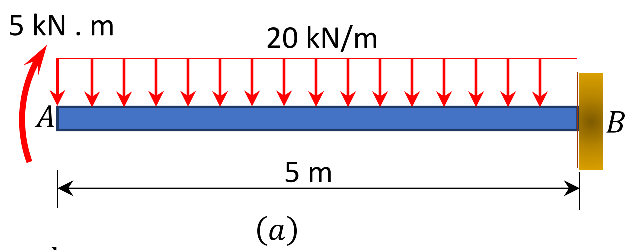
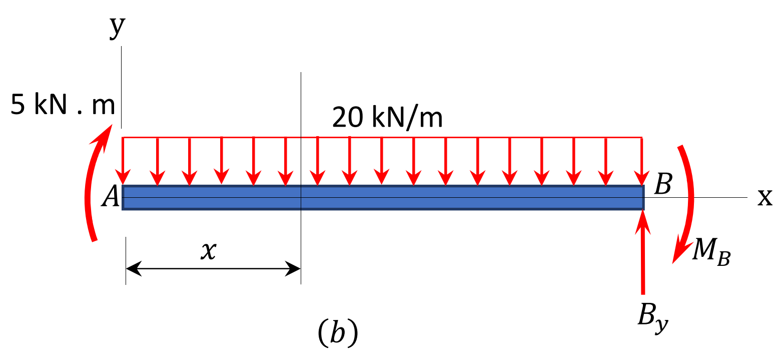


Fig. 7.2. Cantilever beam.



Solution

Equation for bending moment. Passing a section at a distance x from the free-end of the beam, as shown in the free-body diagram in Figure 7.2b, and considering the moment to the right of the section suggests the following:

$$M = 5 - \frac{20x^2}{2} \quad (1)$$

Substituting *M* into equation 7.12 suggests the following:

$$EI\frac{d^2y}{dx^2} = 5 - \frac{20x^2}{2} \qquad (2)$$

Equation for slope. Integrating with respect to x suggests the following:

$$EI\frac{dy}{dx} = 5x - \frac{20x^3}{6} + C_1 \qquad (3)$$

Observe that at the fixed end where $x = L$, $\frac{dy}{dx} = 0$; this is referred to as the boundary condition. Applying these boundary conditions to equation 3 suggests the following:

$$0 = 5L - \frac{20L^3}{6} + C_1$$

$$C_1 = -5 \times 5 + \frac{20(5)^3}{6} = 391.67$$

To obtain the following equation of slope, substitute the computed value of C_1 into equation 3 follows:

$$EI\frac{dy}{dx} = 5x - \frac{20x^3}{6} + 391.67 \qquad (4)$$

Equation for deflection. Integrating equation 4 suggests the following:

$$EIy = \frac{5x^2}{2} - \frac{20x^4}{24} + 391.67x + C_2 \qquad (5)$$

At the fixed end $x = L$, $y = 0$. Applying these boundary conditions to equation 5 suggests the following:

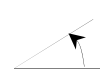
$$0 = \frac{5(L)^2}{2} - \frac{20(L)^4}{24} + 391.67L = \frac{5(5)^2}{2} - \frac{20(5)^4}{24} + 391.67(5) + C_2 = -1500$$

To obtain the following equation of elastic curve, substitute the computed value of C_2 into equation 5, as follows:

$$y = \frac{1}{EI}\left(\frac{5x^2}{2} - \frac{20x^4}{24} + 391.67x - 1500\right) \qquad (6)$$

The slope at the free end, i.e., $\frac{dy}{dx}$ at $x = 0$

$$\left(\frac{dy}{dx}\right)_A = \theta_A = \frac{1}{EI}\left[5(0) - \frac{20(0)^3}{6} + 391.6\right] = \frac{391.67}{EI} \qquad \frac{391.67}{EI}$$



The deflection at the free end, i.e., *y* at $x = 0$

$$y_A = \frac{1}{EI}\left(\frac{5(0)^2}{2} - \frac{20(0)^4}{24} + 391.67(0) - 1500\right) = -\frac{1500}{EI} \qquad \frac{1500}{EI}\downarrow$$

Example 7.2

A simply supported beam AB carries a uniformly distributed load of 2 kips/ft over its length and a concentrated load of 10 kips in the middle of its span, as shown in Figure 7.3a. Using the method of double integration, determine the slope at support A and the deflection at a midpoint C of the beam.

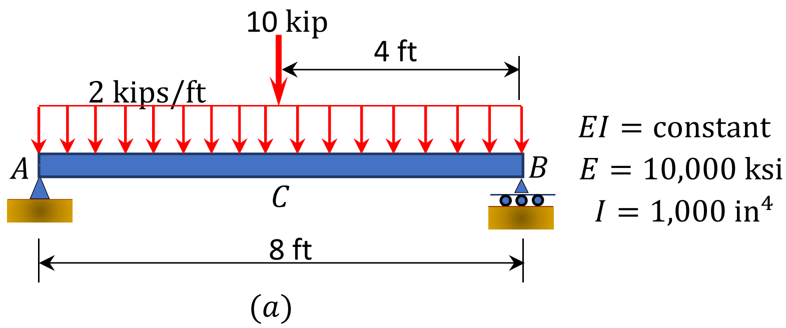
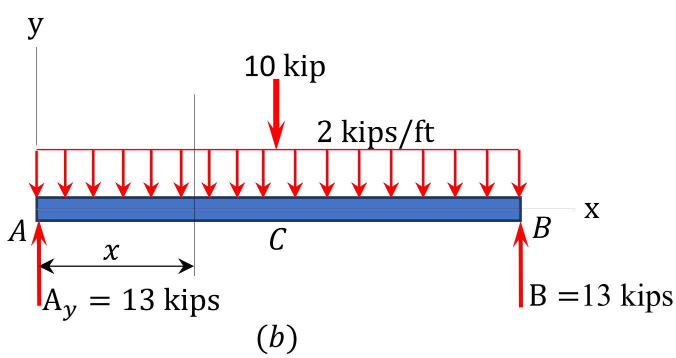


Fig. 7.3. Simply supported beam.



Solution

Support reactions.

$A_y = B_y = \frac{2\times 8}{2} + \frac{10}{2} = 13$ kips by symmetry

Equation for bending moment. The moment at a section of a distance x from support A, as shown in the free-body diagram in Figure 7.3b, is written as follows:

$0 < x < 4$

$$M = 13x - \frac{2x^2}{2} \quad (1)$$

Substituting for M into equation 7.12 suggests the following:

$$EI\frac{d^2y}{dx^2} = M = 13x - \frac{2x^2}{2} \qquad (2)$$

Equation for slope. Integrating equation 2 with respect to x suggests the following:

$$EI\frac{dy}{dx} = \frac{13x^2}{2} - \frac{2x^3}{6} + C_1 \qquad (3)$$

The constant of integration C_1 is evaluated by considering the boundary condition.

At $x = \frac{L}{2}, \frac{dy}{dx} = 0$

Applying the afore-stated boundary conditions to equation 3 suggests the following:

$$0 = \frac{13(4)^2}{2} - \frac{2(4)^3}{6} + C_1$$

$C_1 = -82.67.$

Bringing the computed value of C_1 back into equation 3 suggests the following:

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{13x^2}{2} - \frac{2x^3}{6} - 82.67\right) \qquad (4)$$

Equation for deflection. Integrating equation 4 suggests the following:

$$EIy = \frac{13x^3}{6} - \frac{2x^4}{24} - 82.67x + C_2 \qquad (5)$$

The constant of integration C_2 is evaluated by considering the boundary condition.

At $x = 0, y = 0$

$0 = 0 - 0 - 0 + C_2$

$C_2 = 0$

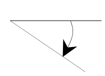
Carrying the computed value of C_2 back into equation 5 suggests the following equation of elastic curve:

$$EIy = \frac{13x^3}{6} - \frac{2x^4}{24} - 82.67x \qquad (6)$$

The slope at A, i.e., $\frac{dy}{dx}$ at $x = 0$

$$\left(\frac{dy}{dx}\right)_A = \theta_A = \frac{1}{EI}\left(\frac{13(0)^2}{2} - \frac{2(0)^3}{6} - 82.67\right) = -\frac{82.67}{EI} = -\frac{82.67}{(10{,}000)(12)^2\left(\frac{1000}{(12)^4}\right)}$$

$$= -0.0012 \text{ rad}$$



Deflection at midpoint C, i.e., at $x = \frac{L}{2}$

$$y_c = \frac{1}{EI}\left[\frac{13(4)^3}{6} - \frac{2(4)^4}{24} - 82.67(4)\right] = -\frac{213.35}{EI} = -\frac{213.35}{(10{,}000)(144)(1000)(12^{-4})}$$

$$= -0.0031\text{ft} = -0.04 \text{ in} \downarrow$$

Example 7.3

A beam carries a distributed load that varies from zero at support A to 50 kN/m at its overhanging end, as shown in Figure 7.4a. Write the equation of the elastic curve for segment AB of the beam, determine the slope at support A, and determine the deflection at a point of the beam located 3 m from support A.

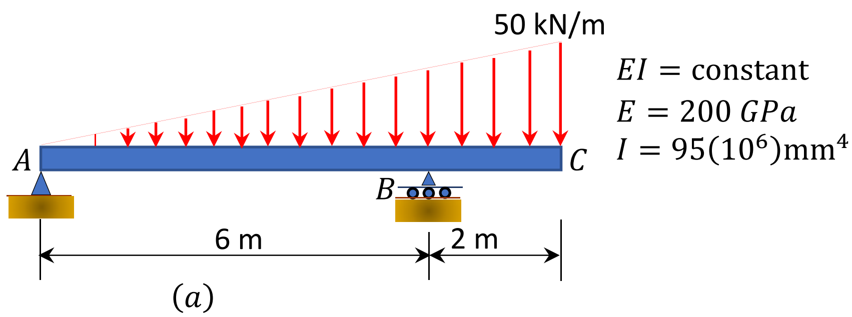
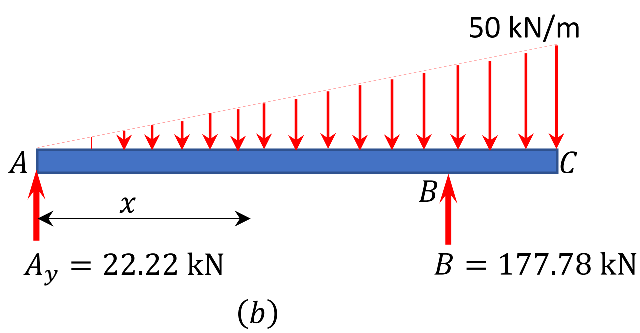


Fig. 7.4. Beam.



Solution

Support reactions. To determine the reactions of the beam, apply the equations of equilibrium, as follows:

$$+\circlearrowleft \sum M_A = 0$$

$-\left(\frac{1}{2}\right)(8)(50)\left(\frac{2}{3}\right)(8) + B_y(6) = 0$

$B_y = 177.78$ kN $\qquad B_y = 177.78 \uparrow$

$+\rightarrow \sum F_x = 0 \quad A_x = 0 \qquad A_x = 0$

$+\uparrow \sum F_y = 0$

$177.78 + A_y - \left(\frac{1}{2}\right)(8)(50) = 0$

$A_y = 22.22$ kN $\qquad A_y = 22.225$ kN $\uparrow$

Equation for bending moment. The moment at a section of a distance x from support A, as shown in the free-body diagram in Figure 7.4b, is as follows:

$0 < x < 6$

$$M = 22.22x - \left(\frac{1}{2}\right)(x)\left(\frac{50x}{8}\right)\left(\frac{x}{3}\right) = 22.22x - \frac{25x^3}{24} \tag{1}$$

Substituting for M into equation 7.12 suggests the following:

$$EI\frac{d^2y}{dx^2} = M = 2.22x - \frac{25x^3}{24} \tag{2}$$

Equation for slope. Integrating equation 2 with respect to x suggests the following:

$$EI\frac{dy}{dx} = \frac{22.22x^2}{2} - \frac{25x^4}{4\times 24} + C_1 \tag{3}$$

Equation for deflection. Integrating equation 3 suggests the equation of deflection, as follows:

$$EIy = \frac{22.22x^3}{6} - \frac{25x^5}{5\times 4\times 24} + C_1x + C_2 \tag{4}$$

To evaluate the constants of integrations, apply the following boundary conditions to equation 4:

At $x = 0, y = 0$

$0 = 0 - 0 + 0 + C_2$

$C_2 = 0$

At $x = 6$ m, $y = 0$

$$0 = \frac{22.22(6)^3}{6} - \frac{25(6)^5}{5\times 4\times 24} + 6C_1$$

$C_1 = -65.82$

Equation of elastic curve.

The equation of elastic curve can now be determined by substituting C_1 and C_2 into equation 4.

$$EIy = \frac{22.22x^3}{6} - \frac{25x^5}{5\times4\times24} - 65.82x$$

To obtain the equations of slope and deflection, substitute the computed value of C_1 and C_2 back into equations 3 and 4:

Equation of slope.

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{22.22x^2}{2} - \frac{25x^4}{96} - 65.82\right) \qquad (5)$$

Equation of deflection.

$$y = \frac{1}{EI}\left(\frac{22.22x^3}{6} - \frac{25x^5}{480} - 65.82x\right) \qquad (6)$$

The slope at A, i.e., $\frac{dy}{dx}$ at $x = 0$

$$\left(\frac{dy}{dx}\right)_A = \theta_A = -\frac{65.82}{EI} = -\frac{65.82}{200(10^6)(95)(10^{-6})} = -0.0035 \text{ rad} \qquad 0.0035 \text{ rad}$$

Deflection at $x = 3$ m from support A.

$$y_x = 3 \text{ m} = -\frac{110.13}{EI} = -0.0058 \text{ m} = -5.80 \text{ mm} \qquad 5.80 \text{ mm} \downarrow$$

7.4 Deflection by Method of Singularity Function

In cases where a beam is subjected to a combination of distributed loads, concentrated loads, and moments, using the method of double integration to determine the deflections of such beams is really involving, since various segments of the beam are represented by several moment functions, and much computational efforts are required to find the constants of integration. Using the method of singularity function in such cases to determine deflections is comparatively easier and relatively quick. This method of analysis was first introduced by Macaulay in 1919, and it entails the use of one equation that contains a singularity or half-range function to describe the entire beam deflection curve. A singularity or half-range function is defined as follows:

$$\langle x-a\rangle^n = \begin{cases} 0 \text{ for } (x-a) < 0 \text{ or } x < a \\ (x-a)^n \text{ for } x-a \geq 0 \text{ or } x \geq a \end{cases}$$

where

x = coordinate position of a point along the beam.
a = any location along the beam where discontinuity due to bending occurs.
n = the exponential values of the functions; this must always be greater than or equal to zero for the functions to be valid.

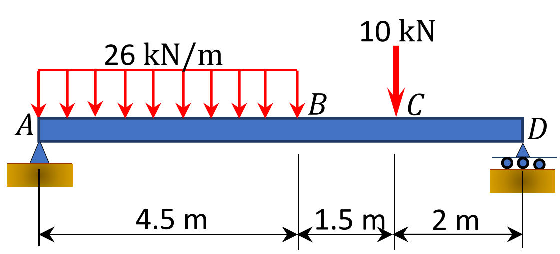
The above outlined definition implies that the quantity $(x - a)$ equals zero or vanishes if it is negative, but it is equal to $(x - a)$ if it is positive.

Procedure for Analysis by Singularity Function Method

- Sketch the free-body diagram of the beam and establish the x and y coordinates.
- Calculate the support reactions and write the moment equation as a function of the x coordinate. The sign convention for the moment is the same as in section 4.3.
- Substitute the moment expression into the equation of the elastic curve and integrate once to obtain the slope. Integrate again to obtain the deflection in the beam.
- Using the boundary conditions, determine the integration constants and substitute them into the equations obtained in step 3 to obtain the slope and the deflection of the beam. A positive slope is counterclockwise and a negative slope is clockwise, while a positive deflection is upward and a negative deflection is downward.
- When computing the slope or deflection at any point on the beam, discard the quantity $(x - a)$ from the equation for slope or deflection if it is negative. If $(x - a)$ is positive, it remains in the equation.

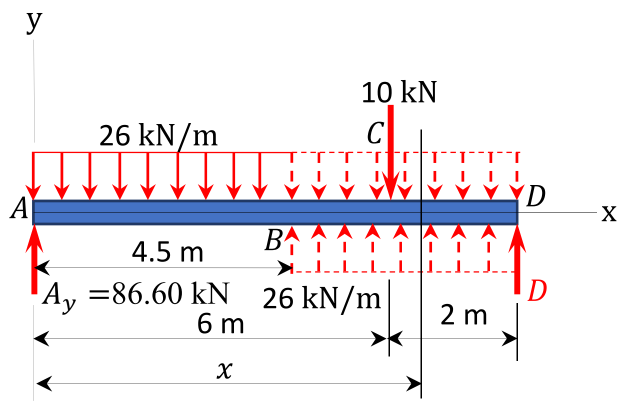
Example 7.4

A simply supported beam is subjected to the combined loading shown in Figure 7.5a. Using the method of singularity function, determine the slope at support A and the deflection at B.



(a)

Fig. 7.5. Simply supported beam.



(b)

Solution

Support reactions. To determine the reaction at support A of the beam, apply the equations of equilibrium, as follows:

$$+\circlearrowleft \sum M_D = 0$$

$$26(4.5)\left(8-\frac{4.5}{2}\right)+10(2)-8A_y = 0$$

$$A_y = 86.6 \text{ kN}$$

Bending moment. Replacing the given distributed load by two equivalent open-ended loadings, as shown in Figure 7.5b, the bending moment at a section located at a distance x from the left support A can be expressed as follows:

$$M = 86.6x-\frac{26x^2}{2}+\frac{26\langle x-4.5\rangle^2}{2}-10\langle x-6\rangle \quad (1)$$

Equation of the elastic curve. Substituting for $M(x)$ from equation 1 into equation 7.12 suggests the following:

$$EI\frac{d^2y}{dx^2} = M = 86.6x-\frac{26x^2}{2}+\frac{26\langle x-4.5\rangle^2}{2}-10\langle x-6\rangle^1 \quad (2)$$

Integrating equation 2 twice suggests the following:

$$EI\frac{dy}{dx} = \frac{86.6x^2}{2}-\frac{26x^3}{3\times2}+\frac{26\langle x-4.5\rangle^3}{3\times2}-\frac{10\langle x-6\rangle^2}{2}+C_1 \quad (3)$$

$$EIy = \frac{86.6x^3}{3\times2}-\frac{26x^4}{4\times3\times2}+\frac{26\langle x-4.5\rangle^4}{4\times3\times2}-\frac{10\langle x-6\rangle^3}{3\times2}+C_1x+C_2 \quad (4)$$

Boundary conditions and computation of constants of integration. Applying the boundary conditions [$x = 0$, $y = 0$] to equation 4 and noting that each bracket contains a negative quantity and, thus, is equal zero by the singularity definition suggests that $C_2 = 0$.

$$0 = 0-0+0-0+C_2$$

$C_2 = 0$

Again, applying the boundary conditions [$x = 8$, $y = 0$] to equation 4 and noting that each bracket contains a positive quantity suggests that the value of the constant C_1 is as follows:

$$0 = \frac{86.6(8)^3}{3 \times 2} - \frac{26(8)^4}{4 \times 3 \times 2} + \frac{26(8-4.5)^4}{4 \times 3 \times 2} - \frac{10(8-6)^3}{3 \times 2} + 8C_1$$

$C_1 = -387.72$

Substituting the values for C_1 and C_2 into equation 4 suggets that the expression for the elastic curve of the beam is as follows:

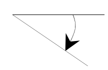
$$y = \frac{1}{EI}\left\{\frac{86.6x^3}{6} - \frac{26x^4}{24} + \frac{26\langle x-4.5\rangle^4}{24} - \frac{10\langle x-6\rangle^3}{6} - 387.72x\right\} \quad (5)$$

Similarly, substituting the values for C_1 into equation 3 suggests the expression for the slope is as follows:

$$\frac{dy}{dx} = \frac{1}{EI}\left\{\frac{86.6x^2}{2} - \frac{26x^3}{6} + \frac{26\langle x-4.5\rangle^3}{6} - \frac{10\langle x-6\rangle^2}{2} - 387.72\right\} \quad (6)$$

The slope at A, i.e., $\frac{dy}{dx}$ at $x = 0$

$$\left(\frac{dy}{dx}\right)_A = \theta_A = -\frac{387.72}{EI}$$

$\frac{387.72}{EI}$ 

The deflection at $x = 4.5$ m from support A

$$y_x = 4.5\text{ m} = \frac{1}{EI}\left\{\frac{86.6(4.5)^3}{6} - \frac{26(4.5)^4}{24} + \frac{26\langle 4.5-4.5\rangle^4}{24} - \frac{10\langle 4.5-6\rangle^3}{6} - 387.72(4.5)\right\}$$

$$y_x = 4.5\text{ m} = -\frac{873.74}{EI}$$

$\frac{873.74}{EI} \downarrow$

Example 7.5

A cantilever beam is loaded with a uniformly distributed load of 4 kips/ft, as shown in Figure 7.6a. Using the method of singularity function, determine the equation of the elastic curve of the beam, the slope at the free end, and the deflection at the free end.

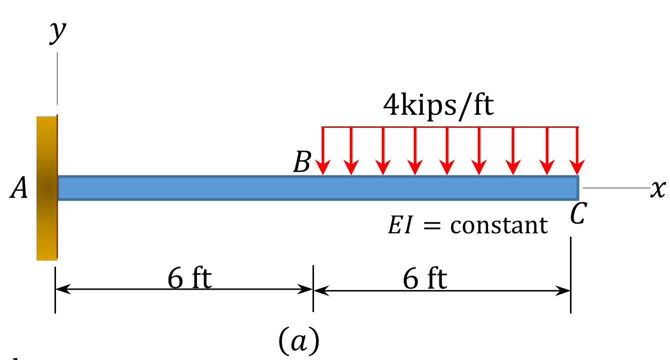
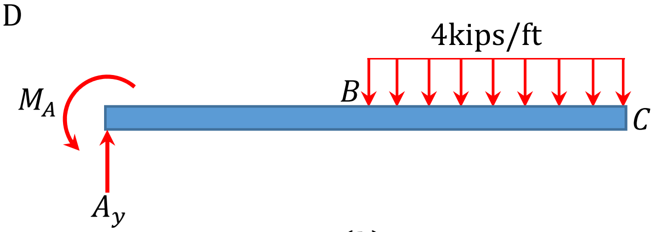


Fig. 7.6. Cantilever beam.

FBD



(b)

Solution

Support reactions. To determine the reaction at support A of the beam, apply the equation of equilibrium, as follows:

$+\circlearrowleft \sum M_A = 0 \qquad M_A - 4(6)(9) = 0 \qquad M_A = 216$ k-ft. $\circlearrowleft$

$+\uparrow \sum F_y = 0 \qquad A_y - 4(6) = 0 \qquad A_y = 24$ k $\uparrow$

$+\rightarrow \sum F_x = 0 \qquad A_x = 0 \qquad A_x = 0$

Bending moment. The bending moment at a section located at a distance x from the fixed end of the beam, shown in Figure 7.6b, can be expressed as follows:

$$M = 24x - 216 - \frac{4\langle x-6\rangle^2}{2} \tag{1}$$

Equation of the elastic curve. Substituting for $M(x)$ from equation 1 into equation 7.12 suggests the following:

$$EI\frac{d^2y}{dx^2} = M = 24x - 216 - \frac{4\langle x-6\rangle^2}{2} \tag{2}$$

Integrating equation 2 twice suggests the following:

$$EI\frac{dy}{dx} = \frac{24(x)^2}{2} - 216x - \frac{4\langle x-6\rangle^3}{6} + C_1 \tag{3}$$

$$EIy = \frac{24(x)^3}{6} - \frac{216(x)^2}{2} - \frac{4\langle x-6\rangle^4}{24} + C_1x + C_2 \tag{4}$$

Boundary conditions and computation of constants of integration. Applying the boundary conditions $[x = 0, \frac{dy}{dx} = 0]$ to equation 3 and noting that the term with a bracket contains a negative quantity and, thus, is equal to zero by the singularity function definition suggests that $C_1 = 0$.

$$\frac{24(0)^2}{2} - 216(0) - \frac{4\langle 0-6\rangle^3}{6} + C_1 = 0 \qquad C_1 = 0$$

Applying the boundary conditions [$x = 0$, $y = 0$] to equation 4 and noting that the term with a bracket contains a negative quantity and, thus, is equal to zero by the singularity function definition suggests that $C_2 = 0$.

$$\frac{24(0)^3}{6} - \frac{216(0)^2}{2} - \frac{4\langle 0-6\rangle^4}{24} + C_1(0) + C_2 = 0 \qquad C_2 = 0$$

To find the elastic curve of the beam, substitute the values for C_1 and C_2 into equation 4, as follows:

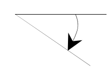
$$y = \frac{1}{EI}\left[\frac{24(x)^3}{6} - \frac{216(x)^2}{2} - \frac{4\langle x-6\rangle^4}{24}\right] \tag{5}$$

Similarly, to find the expression for the slope, substitute the values for C_1 into equation 3, as follows:

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{24(x)^2}{2} - 216x - \frac{4\langle x-6\rangle^3}{6}\right] \tag{6}$$

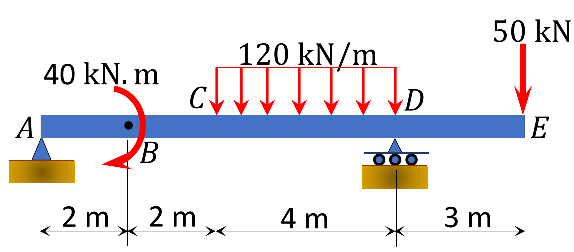
$$\left(\frac{dy}{dx}\right)_C = \theta_C = \frac{1}{EI}\left[\frac{24(12)^2}{2} - 216(12) - \frac{4\langle 12-6\rangle^3}{6}\right] = -\frac{1008}{EI} \qquad \frac{1008}{EI}$$

$$y_C = \frac{1}{EI}\left[\frac{24(12)^3}{6} - \frac{216(12)^2}{2} - \frac{4\langle 12-6\rangle^4}{24}\right] = \frac{-8856}{EI} \qquad \frac{8856}{EI}\downarrow$$



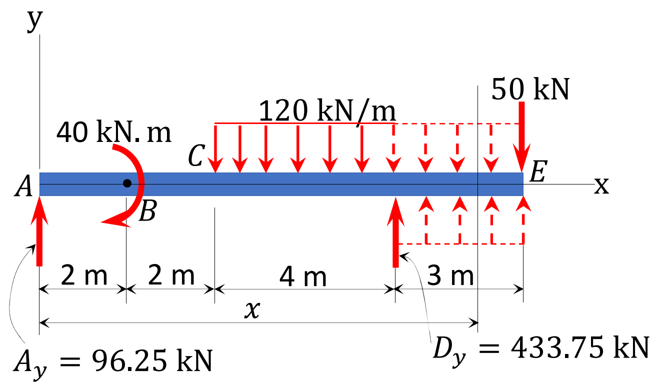
Example 7.6

A beam with an overhang is subjected to a combined loading, as shown in Figure 7.7a. Using the method of the singularity function, determine the slope at support A and the deflection at B.



(a)

Fig. 7.7. Beam with overhang.



(b)

Solution

Support reactions. To determine the reaction at support A of the beam, apply the equations of equilibrium, as follows:

$$+\curvearrowleft \sum M_A = 0$$

$$-40 - 120(4)(6) - 50(11) + 8D_y = 0$$

$D_y = 433.75$ kN $\qquad\qquad D_y = 433.75$ kN ↑

$$+\uparrow \sum F_y = 0$$

$$A_y + 433.75 - 120(4) - 50 = 0$$

$A_y = 96.25$ kN $\qquad\qquad A_y = 96.25$ kN ↑

$+\rightarrow \sum F_x = 0 \quad A_x = 0 \qquad\qquad A_x = 0$

Bending moment. By replacing the given distributed load by two equivalent open-ended loadings and modifying the moment term, as shown in Figure 7.7b, the bending moment at a section located at a distance x from the left support A can be expressed as follows:

$$M = 96.25x + 40\langle x-2\rangle^0 - \frac{120\langle x-4\rangle^2}{2} + \frac{120\langle x-8\rangle^2}{2} + 433.75\langle x-8\rangle \tag{1}$$

Equation of the elastic curve. Substituting for $M(x)$ from equation 1 into equation 7.12 suggests the following:

$$EI\frac{d^2y}{dx^2} = M = 96.25x + 40\langle x-2\rangle^0 - \frac{120\langle x-4\rangle^2}{2} + \frac{120\langle x-8\rangle^2}{2} + 433.75\langle x-8\rangle \tag{2}$$

Integrating equation 2 twice suggests the following:

$$EI\frac{dy}{dx} = \frac{96.25x^2}{2} + 40\langle x-2\rangle^1 - \frac{120\langle x-4\rangle^3}{3\times 2} + \frac{120\langle x-8\rangle^3}{3\times 2} + \frac{433.75\langle x-8\rangle^2}{2} + C_1 \quad (3)$$

$$EIy = \frac{96.25x^3}{3\times 2} + \frac{40\langle x-2\rangle^2}{2} - \frac{120\langle x-4\rangle^4}{4\times 3\times 2} + \frac{120\langle x-8\rangle^4}{4\times 3\times 2} + \frac{433.75\langle x-8\rangle^3}{3\times 2} + C_1x + C_2 \quad (4)$$

Boundary conditions and computation of constants of integration. Applying the boundary conditions [$x = 0$, $y = 0$] to equation 4 and noting that each bracket contains a negative quantity and, thus, is equal to zero by the singularity definition suggests that $C_2 = 0$.

$$0 = 0 + 0 - 0 + 0 + 0 + 0 + C_2$$

$$C_2 = 0$$

Again, applying the boundary conditions [$x = 8$m, $y = 0$] to equation 4 and noting that each bracket contains a positive quantity suggests that the value of the constant C_1 is as follows:

$$0 = \frac{96.25(8)^3}{3\times 2} + \frac{40\langle 8-2\rangle^2}{2} - \frac{120\langle 8-4\rangle^4}{4\times 3\times 2} + \frac{120\langle 8-8\rangle^4}{4\times 3\times 2} + \frac{433.75\langle 8-8\rangle^3}{3\times 2} + 8C_1$$

$$C_1 = -956.67$$

Substituting the values for C_1 and C_2 into equation 4 suggests that the expression for the elastic curve of the beam is as follows:

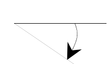
$$y = \frac{1}{EI}\left\{\frac{96.25x^3}{3\times 2} + \frac{40\langle x-2\rangle^2}{2} - \frac{120\langle x-4\rangle^4}{4\times 3\times 2} + \frac{120\langle x-8\rangle^4}{4\times 3\times 2} + \frac{433.75\langle x-8\rangle^3}{3\times 2} - 956.67x\right\}$$

Similarly, substituting the values for C_1 into equation 3 suggests that the expression for the slope is as follows:

$$\frac{dy}{dx} = \frac{1}{EI}\left\{\frac{96.25x^2}{2} + 40\langle x-2\rangle^1 - \frac{120\langle x-4\rangle^3}{3\times 2} + \frac{120\langle x-8\rangle^3}{3\times 2} + \frac{433.75\langle x-8\rangle^2}{2} - 956.67\right\}$$

The slope at A, i.e., $\frac{dy}{dx}$ at $x = 0$

$$\left(\frac{dy}{dx}\right)_A = \theta_A = -\frac{956.67}{EI} \qquad \frac{956.67}{\text{EI}}$$



The deflection at $x = 2$ m from support A

$$y_x = 2\text{ m} = \frac{1}{EI}\left\{\frac{96.25(2)^3}{6} + 0 - 0 + 0 + 0 - 956.67(2)\right\}$$

$$y_x = 2\text{ m} = -\frac{1785}{\text{EI}} \qquad \frac{1785}{EI}\downarrow$$

7.5 Deflection by Moment-Area Method

The moment-area method uses the area of moment divided by the flexural rigidity (*M/EI*) diagram of a beam to determine the deflection and slope along the beam. There are two theorems used in this method, which are derived below.

7.5.1 First Moment-Area Theorem

To derive the first moment-area theorem, consider a portion *AB* of an elastic curve of the deflected beam shown in Figure 7.8b. The beam has a radius of curvature *R*. Figure 7.8c represents the bending moment of this portion. According to geometry, the length of the arc *ds*, of the radius *R*, subtending an angle *dθ*, is equal to the product of the radius of curvature and the angle subtend. Therefore,

$$ds = Rd\theta \tag{7.13}$$

Rearranging equation 1 suggests the following:

$$\frac{d\theta}{ds} = \frac{1}{R} \tag{7.14}$$

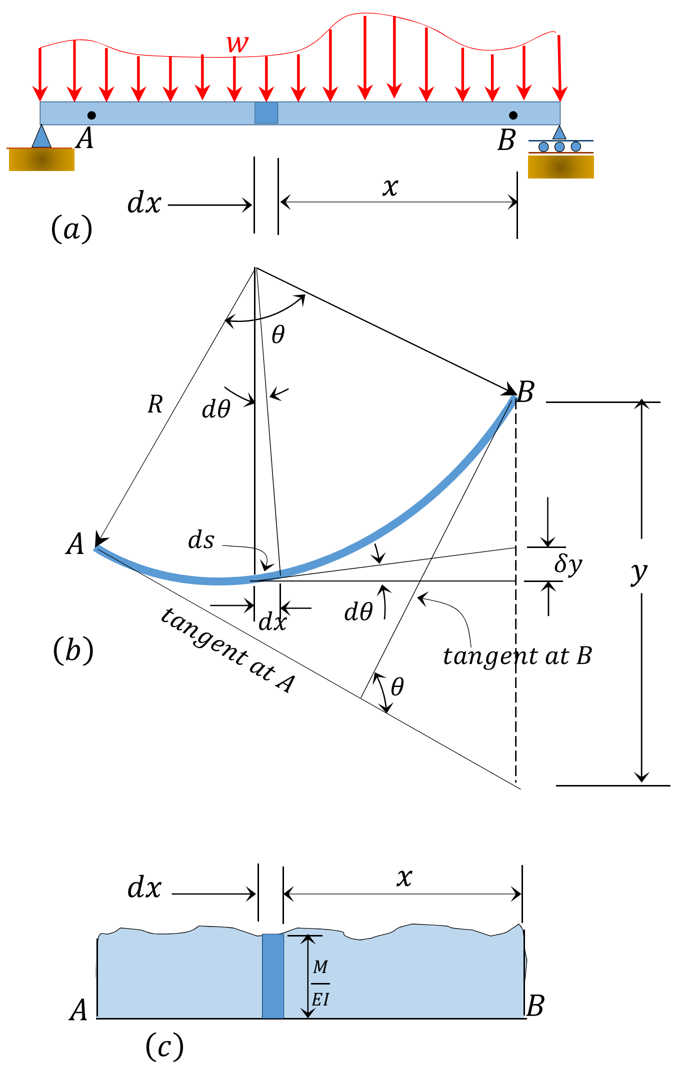


Fig. 7.8. Deflected beam.

Substituting equation 7.14 into equation 7.8 suggests the following:

$$d\theta = \frac{M}{EI}ds \tag{7.15}$$

Since *ds* is infinitesimal because of the small lateral deflection of the beam that is allowed in engineering, it can be replaced by its horizontal projection *dx*. Thus,

$$d\theta = \frac{M}{EI}dx \tag{7.16}$$

The angle θ between the tangents at *A* and *B* can thus be obtained by summing up the subtended angles by the infinitesimal length lying between these points. Thus,

$$\int_A^B d\theta = \int_A^B \frac{M}{EI}dx$$

Or $\theta_{B/A} = \theta_B - \theta_A = \int_A^B \frac{M}{EI}dx$ (7.17)

Equation 7.17 is referred to as the first moment-area theorem. The first moment-area theorem states that the total change in slope between *A* and *B* is equal to the area of the bending moment diagram between these two points divided by the flexural rigidity EI.

7.5.2 Second Moment-Area Theorem

Referring again to Figure 7.8, it is required to determine the tangential deviation of point *B* with respect to point *A*, which is the vertical distance of point *B* from the tangent drawn to the elastic curve at point *A*. To do so, first calculate the contribution $\delta\Delta$ of the element of length *dL* to the vertical distance. According to geometry,

$$\delta y = x d\theta \tag{7.18}$$

Substituting *dθ* from equation 7.15 to equation 7.18 suggests the following:

$$\delta y = \frac{Mx}{EI}dx \tag{7.19}$$

Hence,

$$y = \int_A^B \frac{Mx}{EI}dx \tag{7.20}$$

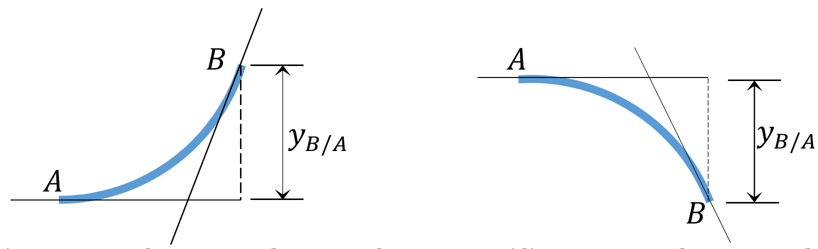
Equation 7.20 is referred to as the second moment area theorem. The second moment-area theorem states that the vertical distance of point *B* on an elastic curve from the tangent to the curve at point *A* is equal to the moment with respect to the vertical through *B* of the area of the bending moment diagram between *A* and *B*, divided by the flexural rigidity, *EI*.

7.5.3 Sign Conventions

The sign conventions for moment-area theorems are as follows:

(1) The tangential deviation of a point *B,* with respect to a tangent drawn at the elastic curve at a point *A*, is positive if *B* lies above the drawn tangent at *A* and negative if it lies below the tangent (see Figure 7.9).

(2) The slope at a point *B*, with respect to a tangent drawn at a point *A* in an elastic curve, is positive if the tangent drawn at *B* rotates in a counterclockwise direction with respect to the tangent at *A* and negative if it rotates in a clockwise direction (see Figure 7.9).



(a) Positive change in slope, and positive deflection.

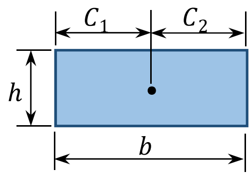
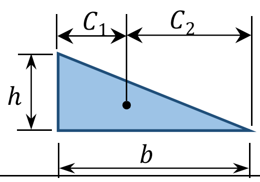
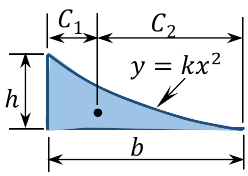
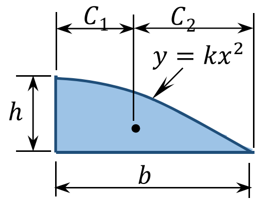
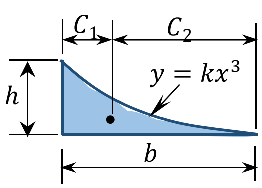
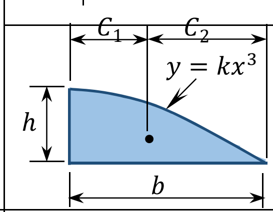
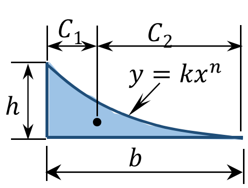
(b) Negative change in slope, and negative deflection.

Fig. 7.9. Sign convention representation.

Procedure for Analysis by Moment-Area Method

- Sketch the free-body diagram of the beam.
- Draw the *M/EI* diagram of the beam. This will look like the conventional bending moment diagram of the beam if the beam is prismatic (i.e. of the same cross section for its entire length).
- To determine the slope at any point, find the angle between a tangent passing the point and a tangent passing through another point on the deflected curve, divide the *M/EI* diagram into simple geometric shapes, and then apply the first moment-area theorem. To determine the deflection or a tangential deviation of any point along the beam, apply the second moment-area theorem.
- In cases where the configuration of the *M/EI* diagram is such that it cannot be divided into simple shapes with known areas and centroids, it is preferable to draw the *M/EI* diagram by parts. This entails introducing a fixed support at any convenient point along the beam and drawing the *M/EI* diagram for each of the applied loads, including the support reactions, prior to the application of any of the theorems to determine what is required.

Table 7.1. Areas and centroids of geometric shapes.

Geometric Shape		Area	Centroid	
			C_1	C_2
Rectangle		bh	$\frac{b}{2}$	$\frac{b}{2}$
Triangle		$\frac{bh}{2}$	$\frac{b}{3}$	$\frac{2b}{3}$
Parabolic spandrel		$\frac{bh}{3}$	$\frac{b}{4}$	$\frac{3b}{4}$
		$\frac{2bh}{3}$	$\frac{3b}{8}$	$\frac{5b}{8}$
Cubic spandrel		$\frac{bh}{4}$	$\frac{b}{5}$	$\frac{4b}{5}$
		$\frac{3bh}{4}$	$\frac{2b}{5}$	$\frac{3b}{5}$
General spandrel		$\frac{bh}{n+1}$	$\frac{b}{n+2}$	$\frac{b(n+1)}{n+2}$

Example 7.7

A cantilever beam shown in Figure 7.10a is subjected to a concentrated moment at its free end. Using the moment-area method, determine the slope at the free end of the beam and the deflection at the free end of the beam. EI = constant.

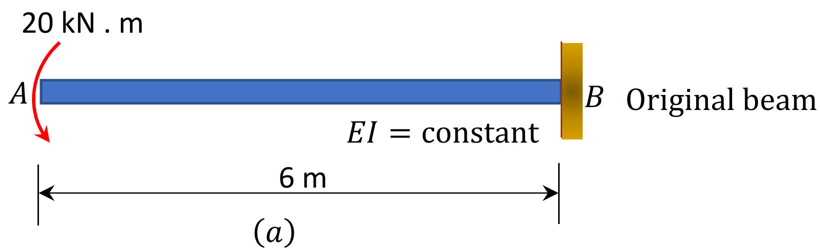
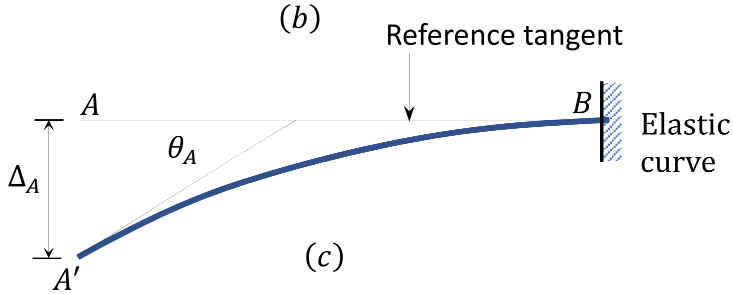


Fig. 7.10. Cantilever beam.

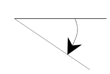


Solution

(M/EI) diagram. First, draw the bending moment diagram for the beam and divide it by the flexural rigidity, EI, to obtain the $\frac{M}{EI}$ diagram shown in Figure 7.10b.

Slope at A. The slope at the free end is equal to the area of the $\frac{M}{EI}$ diagram between A and B, according to the first moment-area theorem. Using this theorem and referring to the $\frac{M}{EI}$ diagram suggests the following:

$$\theta_A = -\left(\frac{1}{EI}\right)(6)(20) = -\frac{120}{EI} \qquad \theta_A = \frac{120}{EI}$$



Deflection at A. The deflection at the free end of the beam is equal to the moment with respect to the vertical through A of the area of the $\frac{M}{EI}$ diagram between A and B, according to the second moment-area theorem. Using this theorem and referring to Figure 7.10b and Figure 7.10c suggests the following:

$\Delta_A = -\left(\frac{1}{EI}\right)(6)(20)(3) = -\frac{360}{EI}$ $\qquad\qquad$ $\Delta_A = \frac{360}{EI} \downarrow$

Example 7.8

A propped cantilever beam carries a uniformly distributed load of 4 kips/ft over its entire length, as shown in Figure 7.11a. Using the moment-area method, determine the slope at A and the deflection at A.

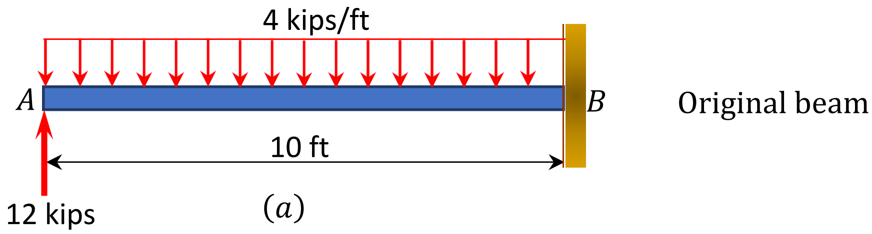
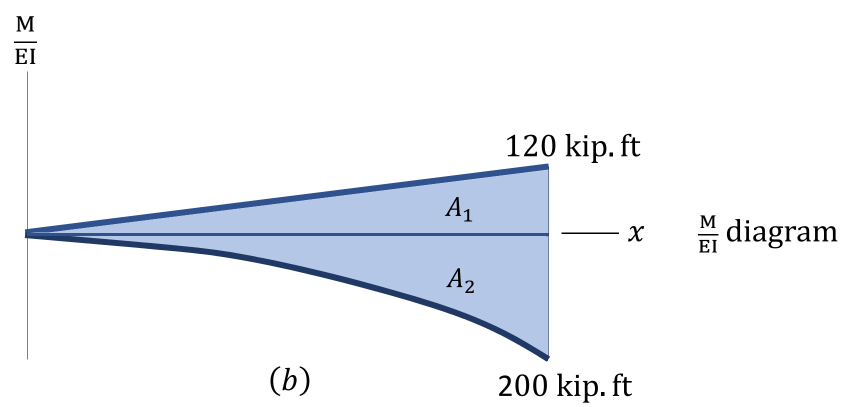
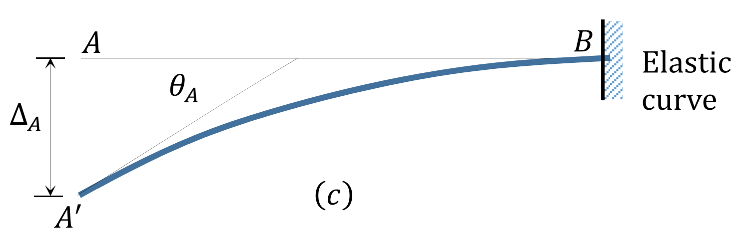


Fig. 7.11. Propped cantilever beam.



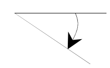


Solution

(*M*/*EI*) diagram. First, draw the bending moment diagram for the beam and divide it by the flexural rigidity, *EI*, to obtain the $\frac{M}{EI}$ diagram shown in Figure 7.11b.

Slope at *A*. The slope at the free end is equal to the area of the $\frac{M}{EI}$ diagram between *A* and *B*. The area between these two points is indicated as A_1 and A_2 in Figure 7.11b. Use Table 7.1 to find the computation of A_2, whose arc is parabolic, and the location of its centroid. Noting from the table that $A = \frac{1}{3}bh$ and applying the first moment-area theorem suggests the following:

$$\theta_A = A_1 - A_2 = \left(\frac{1}{EI}\right)\left(\frac{1}{2}\right)(10)(120) - \left(\frac{1}{EI}\right)\left(\frac{10\times 200}{3}\right) = -\frac{66.67}{EI} \qquad \theta_A = \frac{66.67}{EI}$$



Deflection at *A*. The deflection at *A* is equal to the moment of area of the $\frac{M}{EI}$ diagram between *A* and *B* about *A*. Thus, using the second moment-area theorem and referring to Figure 7.11b and Figure 7.11c suggests the following:

$$\Delta_A = A_1\left(\frac{L}{3}\right) - A_2\left(\frac{3L}{4}\right) = \left(\frac{1}{EI}\right)\left(\frac{1}{2}\right)(10)(120)\left(\frac{2\times 10}{3}\right) - \left(\frac{1}{EI}\right)\left(\frac{10\times 200}{3}\right)\left(\frac{3\times 10}{4}\right) = -\frac{1000}{EI} \qquad \Delta_A = \frac{1000}{EI}\downarrow$$

Example 7.9

A simply supported timber beam with a length of 8 ft will carry a distributed floor load of 500 lb/ft over its entire length, as shown Figure 7.12a. Using the moment area theorem, determine the slope at end *B* and the maximum deflection.

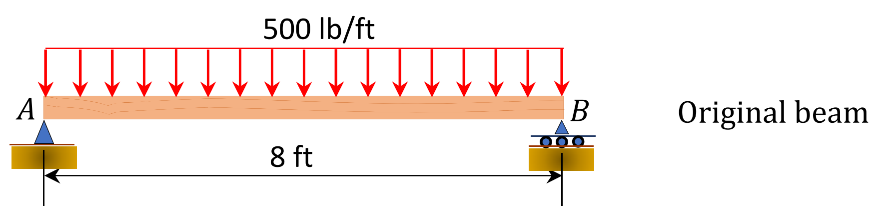
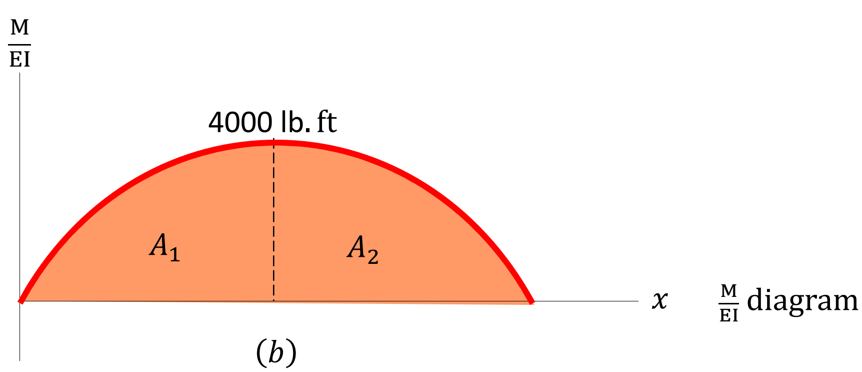
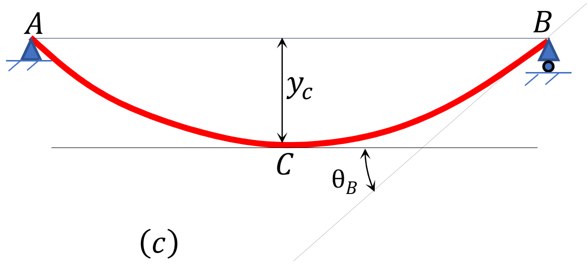


Fig. 7.12. Simply supported timber beam.





(c)

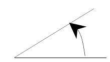
Elastic curve

Solution

(M/EI) diagram. First, draw the bending moment diagram for the beam, and divide it by the flexural rigidity, *EI*, to obtain the $\frac{M}{EI}$ diagram shown in Figure 7.12b.

Slope at *B*. The slope at *B* is equal to the area of the $\frac{M}{EI}$ diagram between *B* and *C*. The area between these two points is indicated as A_2 in Figure 7.12b. Applying the first moment-area theorem suggests the following:

$$\theta_B = A_2 = \left(\frac{1}{EI}\right)\left(\frac{2bh}{3}\right) = \left(\frac{1}{EI}\right)\left(\frac{2(4)(4000)}{3}\right) = \frac{10666.67}{EI} \qquad \theta_B = \frac{10666.67}{EI}$$



Maximum deflection. The maximum deflection occurs at the center of the beam (point C). It is equal to the moment of the area of the $\frac{M}{EI}$ diagram between *B* and *C* about *B*. Thus,

$$\Delta_c = A_2\left(\frac{5b}{8}\right) = \left(\frac{1}{EI}\right)\left(\frac{2(4)(4000)}{3}\right)\left(\frac{5(4)}{8}\right) = \frac{26666.67}{EI} \qquad \Delta_c = \frac{26666.67}{EI} \downarrow$$

Example 7.10

A prismatic timber beam is subjected to two concentrated loads of equal magnitude, as shown in Figure 7.13a. Using the moment-area method, determine the slope at *A* and the deflection at point *C*.

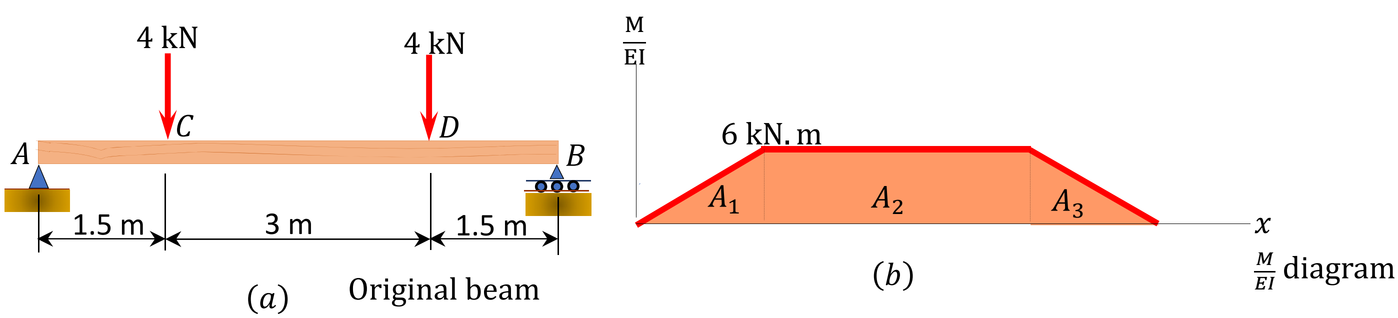
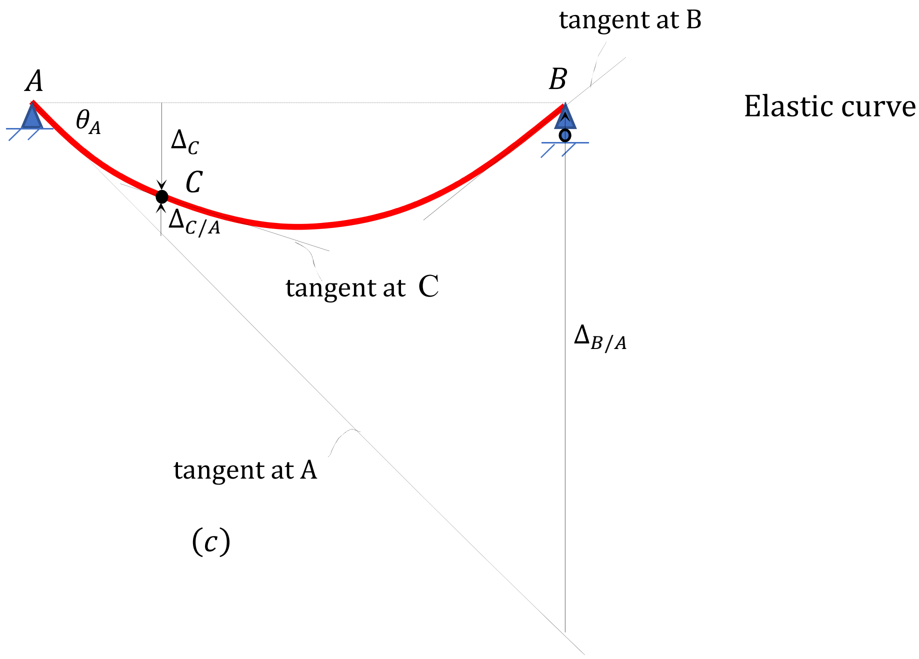


Fig. 7.13. Prismatic timber beam.



(c)

Solution

(M/EI) diagram. First, draw the bending moment diagram for the beam and divide it by the flexural rigidity, EI, to obtain the $\frac{M}{EI}$ diagram shown in Figure 7.13b.

Slope at A. The deflection and the rotation of the beam are small since they occur within the elastic limit. Thus, the slope at support A can be computed using the small angle theorem, as follows:

$$\theta_A = \frac{\Delta_{B/A}}{L} = \frac{\Delta_{B/A}}{6}$$

To determine the tangential deviation of B from A, apply the second moment-area theorem. According to the theorem, it is equal to the moment of the area of the $\frac{M}{EI}$ diagram between A and B about B. Thus,

$$\Delta_{B/A} = A_1\left(1.5 + 3 + \frac{1}{3} \times 1.5\right) + A_2(1.5 + 1.5) + A_3\left(\frac{2}{3} \times 1.5\right)$$

$$\Delta_{B/A} = \frac{1}{EI}\left[\frac{1}{2}(1.5)(6)\left(\frac{2}{3} \times 1.5\right) + (3)(6)(1.5 + 1.5) + \frac{1}{2}(1.5)(6)\left(1.5 + 3 + \frac{1}{3} \times 1.5\right)\right]$$

$$\Delta_{B/A} = \frac{81}{EI}$$

Thus, the slope at A is

$\theta_A = \frac{\Delta_{B/A}}{L} = \frac{81}{6EI} = \frac{13.5}{EI}$ $\theta_A = \frac{13.5}{EI}$

Deflection at *C*. The deflection at *C* can be obtained by proportion.

$$\frac{\Delta_{B/A}}{6} = \frac{\Delta_c + \Delta_{C/A}}{1.5}$$

$$\Delta_c = \frac{(1.5)(\Delta_{B/A})}{6} - \Delta_{C/A}$$

Similarly, the tangential deviation of *C* from *A* can be determined as the moment of the area of the $\frac{M}{EI}$ diagram between *A* and *C* about *C*.

$$\Delta_{C/A} = \frac{1}{EI}\left[\frac{1}{2}(1.5)(6)\left(\frac{2}{3} \times 1.5\right)\right] = \frac{9}{2EI}$$

Therefore, the deflection at *C* is

$\Delta_c = \frac{(1.5)(81)}{6EI} - \frac{9}{2EI} = \frac{15.75}{EI}$ $\Delta_c = \frac{15.75}{EI}$

7.6 Deflection by the Conjugate Beam Method

The conjugate beam method, developed by Heinrich Muller-Breslau in 1865, is one of the methods used to determine the slope and deflection of a beam. The method is based on the principle of statics.

A conjugate beam is defined as a fictitious beam whose length is the same as that of the actual beam, but with a loading equal to the bending moment of the actual beam divided by its flexural rigidity, *EI*.

The conjugate beam method takes advantage of the similarity of the relationship among load, shear force, and bending moment, as well as among curvature, slope, and deflection derived in previous chapters and presented in Table 7.2.

Table 7.2. Relationship between load-shear-bending moment and curvature-slope-deflection.

Load-shear-bending moment	Curvature-slope-deflection
$V(x) = \int w dx$	$\theta(x) = \int \frac{M}{EI} dx$
$M(x) = \int V dx$	$\Delta(x) = \int \theta(x)\, dx$
or $M(x) = \iint w dx\, dx$	or $\Delta(x) = \iint \frac{M}{EI} dx\, dx$

7.6.1 Supports for Conjugate Beams

The supports for conjugate beams are shown in Table 7.3 and the examples of real and conjugate beams are shown in Figure 7.4.

Table 7.3. Supports for conjugate beams.

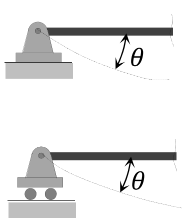
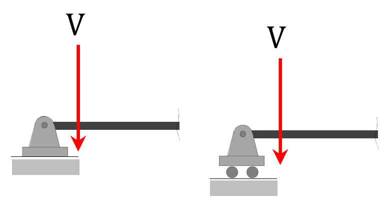
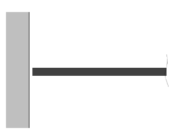
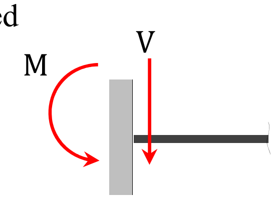
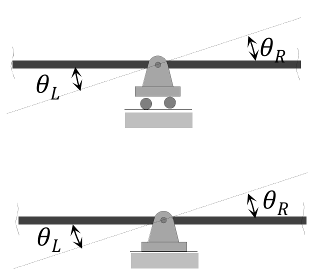
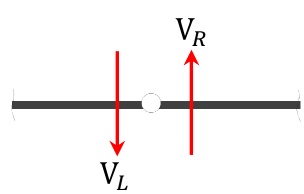
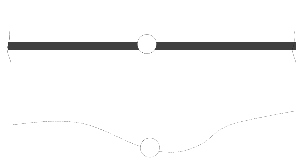
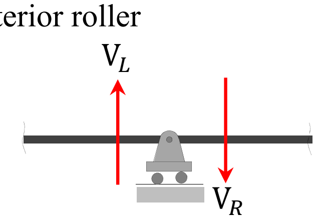
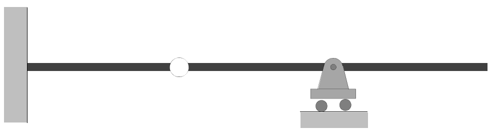
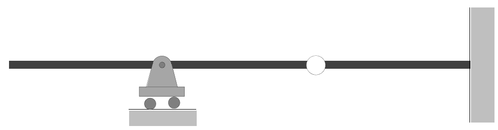
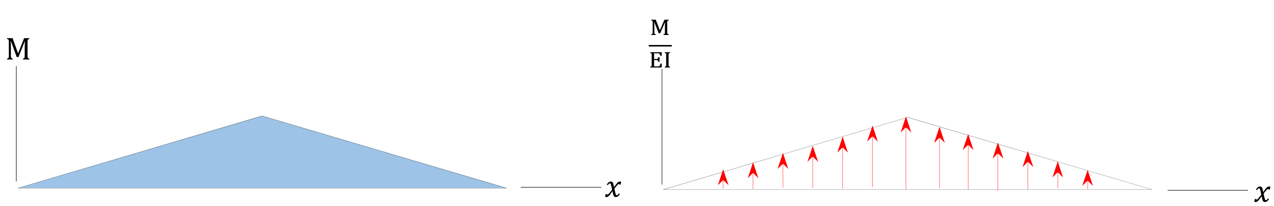
Real Support		Conjugate Support	
Pin or roller	Slope and Deflection	Pin or roller	Shear and Moment
	$\theta \neq 0$ $\Delta = 0$		$V \neq 0$ $M = 0$
Fixed 	$\theta = 0$ $\Delta = 0$	Free end 	$V = 0$ $M = 0$
Free end 	$\theta \neq 0$ $\Delta \neq 0$	Fixed 	$V \neq 0$ $M \neq 0$
Interior pin or roller support 	$\theta \neq 0$ $\Delta = 0$	Internal hinge 	$V \neq 0$ $M = 0$
Internal hinge 	$\theta \neq 0$ $\Delta \neq 0$	Interior roller 	$V \neq 0$ $M \neq 0$

Table 7.4 Real beams and their conjugate.

Real Beam	Conjugate Beam
	
	
	
	
	
	

7.6.2 Sign Convention

For a positive curvature diagram, where there is a positive ordinate of the $\frac{M}{EI}$ diagram, the load in the conjugate should point in the positive y direction (upward) and vice versa (see Figure 7.14).



(a) Real beam diagram for positive moment

(b) Conjugate beam diagram for deflection and slope analysis

Fig. 7.14. Positive curvature diagram.

If the convention stated for positive curvature diagrams is followed, then a positive shear force in the conjugate beam equals the positive slope in the real beam, and a positive moment in the conjugate beam equals a positive deflection (upward movement) of the real beam. This is shown in Figure 7.15.

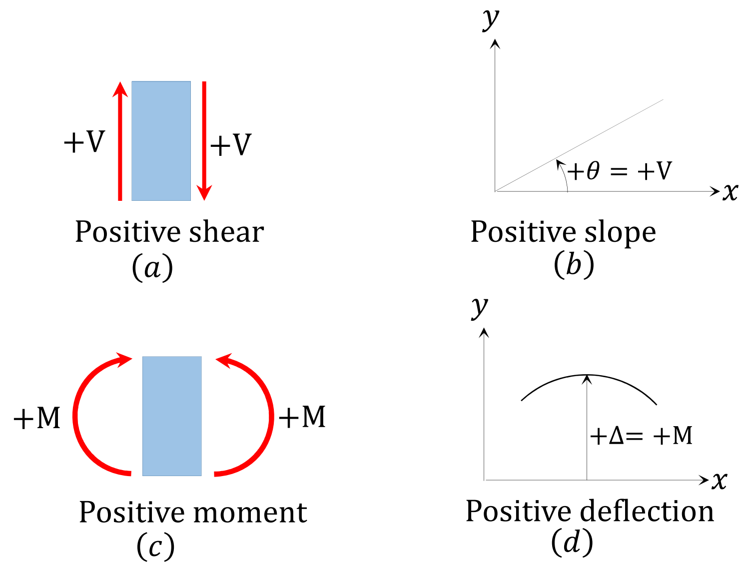


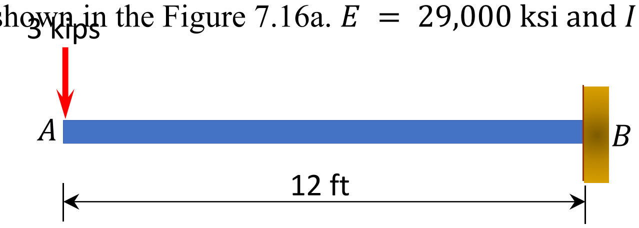
Fig. 7.15. Shear and slope in beam.

Procedure for Analysis by Conjugate Beam Method

- Draw the curvature diagram for the real beam.
- Draw the conjugate beam for the real beam. The conjugate beam has the same length as the real beam. A rotation at any point in the real beam corresponds to a shear force at the same point in the conjugate beam, and a displacement at any point in the real beam corresponds to a moment in the conjugate beam.
- Apply the curvature diagram of the real beam as a distributed load on the conjugate beam.
- Using the equations of static equilibrium, determine the reactions at the supports of the conjugate beam.
- Determine the shear force and moment at the sections of interest in the conjugate beam. These shear forces and moments are equal to the slope and deflection, respectively, in the real beam. Positive shear in the conjugate beam implies a counterclockwise slope in the real beam, while a positive moment denotes an upward deflection in the real beam.

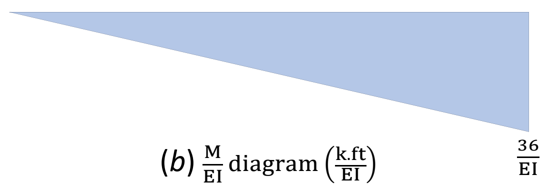
Example 7.11

Using the conjugate beam method, determine the slope and the deflection at point A of the cantilever beam shown in the Figure 7.16a. $E = 29{,}000$ ksi and $I = 280$ in.^{4}

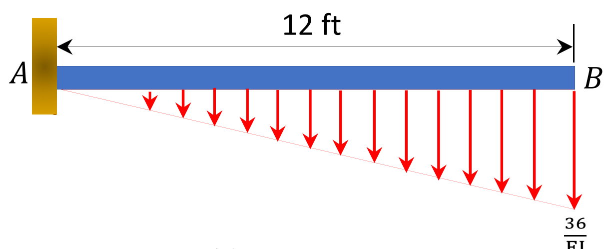


(a) Real Beam

Fig. 7.16. Conjugate beam.



(b) $\frac{M}{EI}$ diagram $\left(\frac{k.ft}{EI}\right)$



(c) Conjugate Beam

Solution

(M/EI) diagram. First, draw the bending moment diagram for the beam and divide it by the flexural rigidity, EI, to obtain the $\frac{M}{EI}$ diagram shown in Figure 7.16b.

Conjugate beam. The conjugate beam loaded with the $\frac{M}{EI}$ diagram is shown in Figure 7.16c. Notice that the free end in the real beam becomes fixed in the conjugate beam, while the fixed end in the real beam becomes free in the conjugate beam. The $\frac{M}{EI}$ diagram is applied as a downward load in the conjugate beam because it is negative in Figure 7.16b.

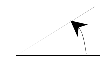
Slope at A. The slope at A in the real beam is the shear at A in the conjugate beam. The shear at A in the conjugate is as follows:

$$V_A = \left(\frac{1}{2}\right)(12)\left(\frac{36}{EI}\right) = \frac{216 \text{ k-ft}^2}{EI}$$

The same sign convention for shear force used in Chapter 4 is used here.

Thus, the slope in the real beam at point A is as follows:

$$\theta_A = \frac{216 \text{ k-ft}^2}{EI} = \frac{216\,(12)^2}{(29{,}000)(280)} = 0.0038 \text{ rad} \qquad = 0.0038 \text{ rad}$$



Deflection at A. The deflection at A in the real beam equals the moment at A of the conjugate beam. The moment at A of the conjugate beam is as follows:

$$M_A = -\left(\frac{1}{2}\right)(12)\left(\frac{36}{EI}\right)\left(\frac{2}{3}\times 12\right) = -\frac{1728 \text{ k-ft}^3}{EI}$$

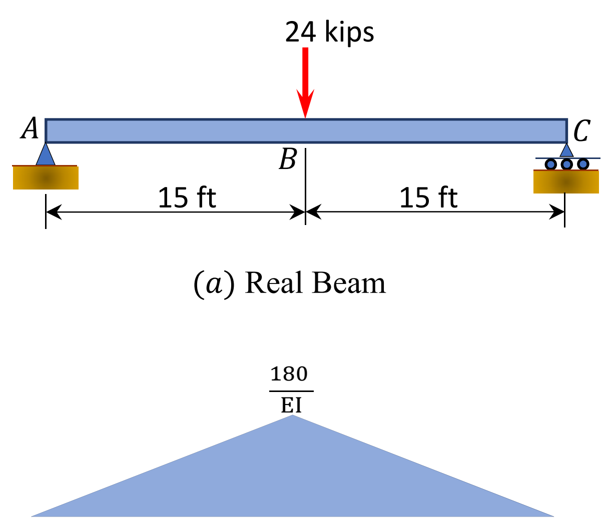
The same sign convention for bending moment used in Chapter 4 is used here.

Thus, the deflection in the real beam at point A is as follows:

$$\Delta_A = -\frac{1728\,(12)^3}{(29{,}000)(280)} = -0.37 \text{ in} \qquad \Delta_A - 0.37 \text{ in} \downarrow$$

Example 7.12

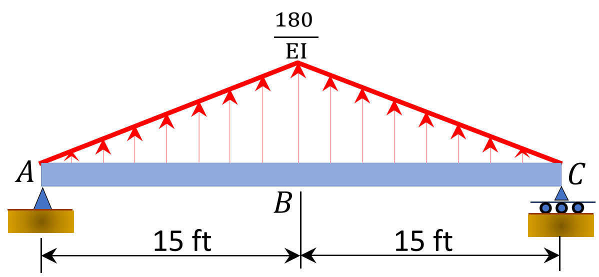
Using the conjugate beam method, determine the slope at support A and the deflection under the concentrated load of the simply supported beam at B shown in Figure 7.17a.
$E = 29{,}000$ ksi and $I = 800$ in.^{4}



(a) Real Beam

$(b)\ \frac{M}{EI}$ diagram $\left(\frac{\text{k.ft}}{\text{EI}}\right)$

Fig. 7.17. Simply supported beam.



(c) Conjugate Beam

Solution

(M/EI) diagram. First, draw the bending moment diagram for the beam and divide it by the flexural rigidity, *EI*, to obtain the moment curvature $(\frac{M}{EI})$ diagram shown in Figure 7.17b.

Conjugate beam. The conjugate beam loaded with the $\frac{M}{EI}$ diagram is shown in Figure 7.17c. Notice that *A* and *C*, which are simple supports in the real beam, remain the same in the conjugate beam. The $\frac{M}{EI}$ diagram is applied as an upward load in the conjugate beam because it is positive in Figure 7.17b.

Reactions for conjugate beam. The reaction at supports of the conjugate beam can be determined as follows:

$$A_y = B_y = -\frac{1}{EI}\left(\frac{1}{2}\right)(30)(180)(0.5) = -\frac{1350\text{ k. ft}^2}{EI} \text{ due to symmetry in loading}$$

Slope at *A*. The slope at *A* in the real beam is the shear force at *A* in the conjugate beam. The shear at *A* in the conjugate beam is as follows:

$$V_A = -\frac{1350\text{ k. ft}^2}{EI}$$

Thus, the slope at support *A* of the real beam is as follows:

$$\theta_A = -\frac{1350\text{ k. ft}^2}{EI} = -\frac{1350(12)^2}{(29{,}000)(800)} = -0.0084\text{ rad} \qquad \theta_A = 0.0084\text{ rad}$$

Deflection at *B*. The deflection at *B* in the real beam equals the moment at *B* of the conjugate beam. The moment at *B* of the conjugate beam is as follows:

$$M_B = \frac{1}{EI}\left[-(1350)(15) + \left(\frac{1}{2}\right)(15)(180)\left(\frac{15}{3}\right)\right] = -\frac{13500\text{ k. ft}^3}{EI}$$

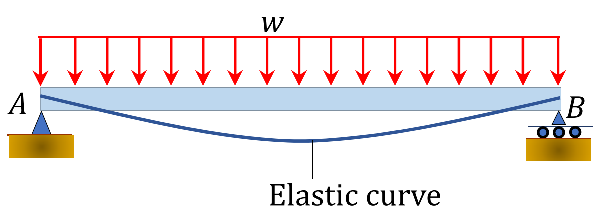
The deflection at B of the real beam is as follows:

$$\Delta_B = -\frac{33750 \text{ k. ft}^3}{EI} = -\frac{13500\,(12)^3}{(29{,}000)(800)} = -1.01 \text{ in.} \qquad \Delta_B = 1.01 \text{ in.}\downarrow$$

Chapter Summary

Deflection of beams through geometric methods: The geometric methods considered in this chapter includes the double integration method, singularity function method, moment-area method, and conjugate-beam method. Prior to discussion of these methods, the following equation of the elastic curve of a beam was derived:

$$EI\left(\frac{d^2y}{dx^2}\right) = M(x)$$



Method of double integration: This method involves integrating the equation of elastic curve twice. The first integration yields the slope, and the second integration gives the deflection. The constants of integration are determined considering the boundary conditions.

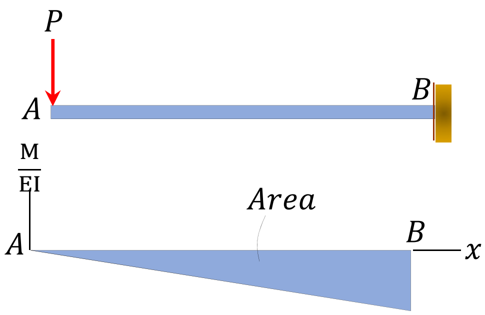
Method of singularity function: This method involves using a singularity or half-range function to describe the equation of the elastic curve for the entire beam. A half-range function can be written in the general form as follows:

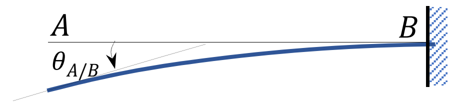
$$\langle x-a\rangle^n = \begin{cases} 0 \; for \; (x-a) < 0 \; or \; x < a \\ (x-a)^n \; for \; x-a \geq 0 \; or \; x \geq a \end{cases}$$

The method of singularity is best suited for beams with many discontinuities due to concentrated loads and moments. The method significantly reduces the number of constants of integration needed to be determined and, thus, makes computation easier when compared with the method of double integration.

Moment-area method: This method uses two theorems to determine the slope and deflection at specified points on the elastic curve of a beam. The two theorems are as follows:

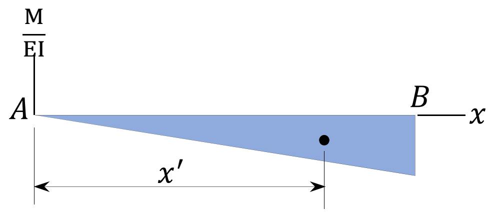
First moment-area theorem: The change in slope between any two points on the elastic curve of a beam equals the area of the $\frac{M}{EI}$ diagram between these two points.

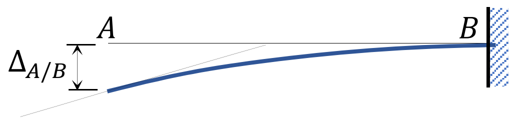




$\theta_{A/B} = Area\ of\ M/EI$ diagram

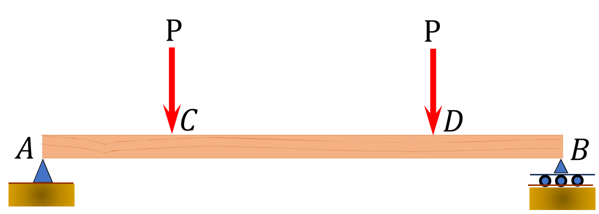
Second moment-area theorem: The vertical deflection of point A from the tangent drawn to the elastic curve at point B equals the moment of the area under the $\frac{M}{EI}$ diagram between these two points about point A.



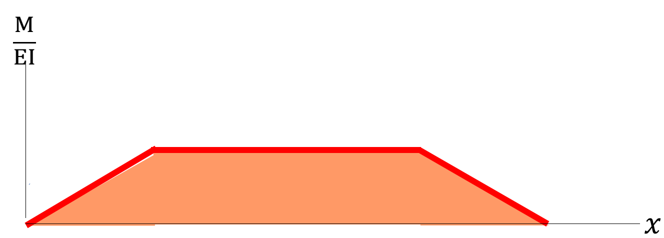


$\Delta_{A/B} = (Area\ of\ M/EI\ \ diagram)(x')$

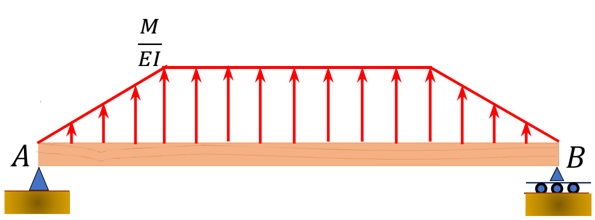
Conjugate beam method: A conjugate beam has been defined as an imaginary beam with the same length as that of the actual beam but with a loading equal the $\frac{M}{EI}$ diagram of the actual beam. The supports in the actual beams are replaced with fictitious supports with boundary conditions that will result in the bending moment and the shear force at a specific point in a conjugate beam equaling the deflection and slope, respectively, at the same points in the actual beam.



Actual beam



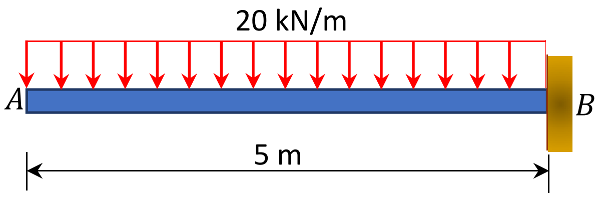
$\frac{M}{EI}$ diagram of actual beam



Conjugate beam

Practice Problems

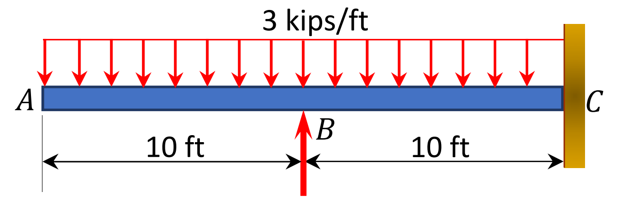
7.1 Using the double integration method, determine the slopes and deflections at the free ends of the cantilever beams shown in Figure P7.1 through Figure P7.4. EI = constant.



$E = 200\ GPa$

$I = 600 \times 10^6\ \text{mm}^4$

Fig. P7.1. Cantilever beam.

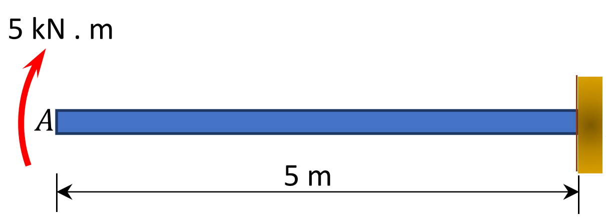


EI = constant

E = 29,000 ksi

$I = 1000\ \text{in}^4$

Fig. P7.2. Cantilever beam.



$E = 200\ GPa$

$I = 400 \times 10^6\ \text{mm}^4$

Fig. P7.3. Cantilever beam.

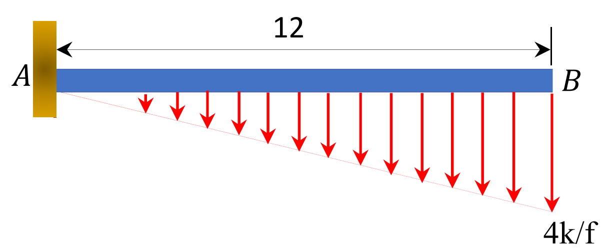


Fig. P7.4. Cantilever beam.

EI = constant

E = 29,000 ksi

$I = 600\ \text{in}^2$

7.2 Using the double integration method, determine the slopes at point A and the deflections at midpoint C of the beams shown in Figure P7.5 and Figure P7.6. EI = constant.

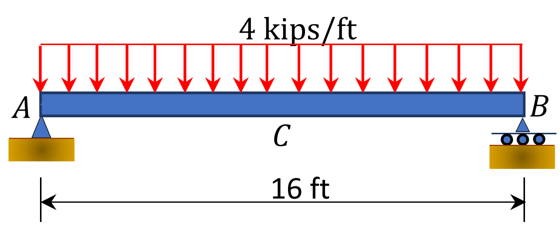
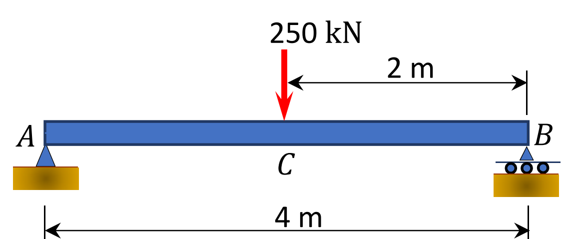


Fig. P7.5. Beam.

EI = constant

E = 10,000 ksi

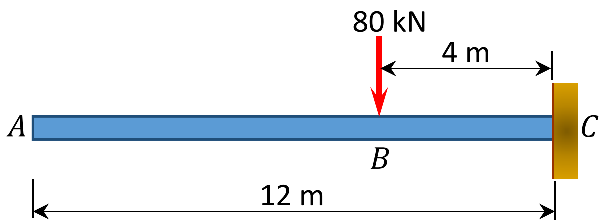
$I = 1{,}000\ \text{in}^4$



$E = 200\ GPa$

Fig. P7.6. Beam. $I = 600 \times 10^6\ \text{mm}^4$

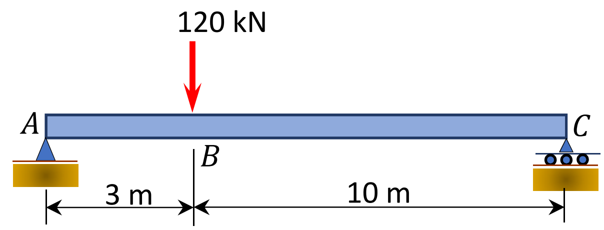
7.3 Using the conjugate beam method, determine the slope at point A and the deflection at point B of the beam shown in Figure P7.7 through Figure P7.10.



$E = 200\ GPa$

$I = 500 \times 10^6\ \text{mm}^4$

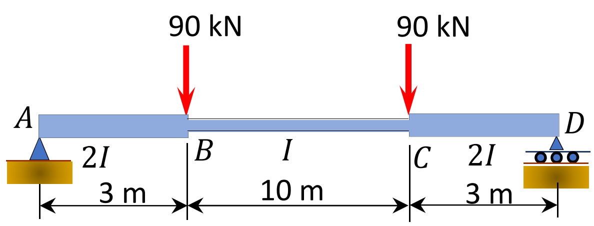
Fig. P7.7. Beam.



$E = 200\ GPa$

$I = 800 \times 10^6\ \text{mm}^4$

Fig. P7.8. Beam.



$E = 200\ GPa$

$I = 800 \times 10^6\ \text{mm}^4$

Fig. P7.9. Beam.

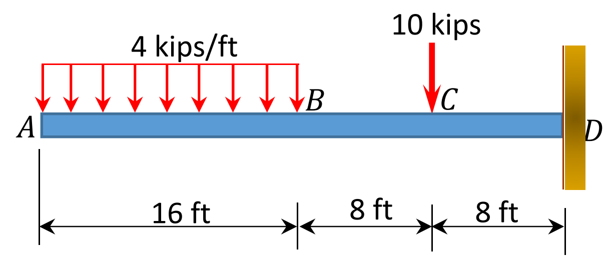


Fig. P7.10. Beam.

$E = 29{,}000$ ksi

$I = 3{,}000\ \text{in}^4$

7.4 Using the moment-area method, determine the deflection at point A of the cantilever beam shown in Figure P7.11 through Figure P7.12.

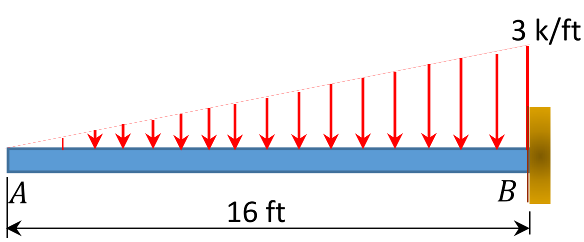


Fig. P7.11. Cantilever beam.

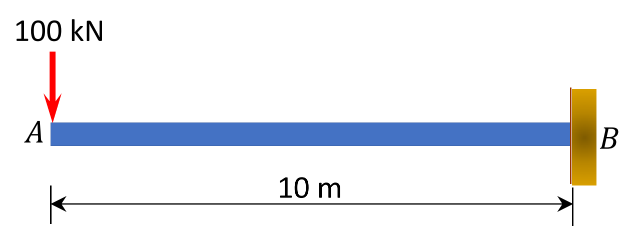
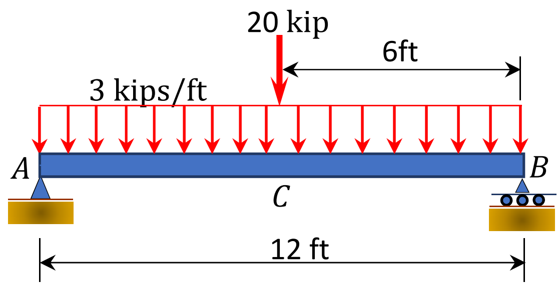


Fig. P7.12. Cantilever beam.

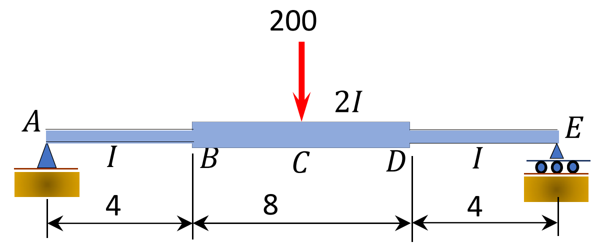
7.5 Using the moment-area method, determine the slope at point A and the slope at the midpoint C of the beams shown in Figure P7.13 and Figure P7.14.



EI = constant

E = 10,000 ksi

Fig. P7.13. Beam. I = 1,000 in^4



$E = 200\ GPa$

$I = 800 \times 10^6\ mm^4$

Fig. P7.14. Beam.

7.6 Using the method of singularity function, determine the slope and the deflection at point A of the cantilever beam shown in Figure P7.15.

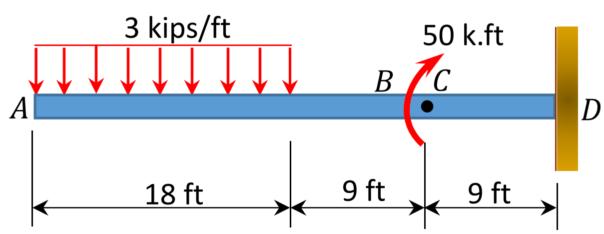


Fig. P7.15. Cantilever beam. E = 29,000 ksi

I = 3,000 in^4

7.7 Using the method of singularity function, determine the slope at point B and the slope at point C of the beam with the overhang shown in Figure P7.16. EI = constant. E = 200 GPa, I = 500×10^6 mm^4.

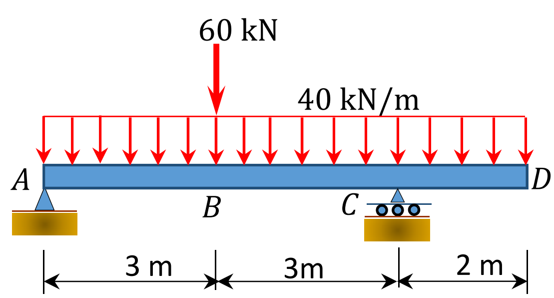


Fig. P7.16. Beam.

7.8 Using the method of singularity function, determine the slope at point C and the deflection at point D of the beam with overhanging ends, as shown in Figure P7.17. EI = constant.

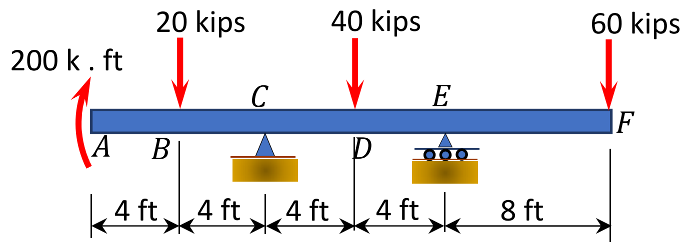


Fig. P7.17. Beam. $E = 29{,}000$ ksi
$I = 3{,}000$ in^4

7.9 Using the method of singularity function, determine the slope at point *A* and the deflection at point *B* of the beam shown in Figure P7.18. *EI* = constant.

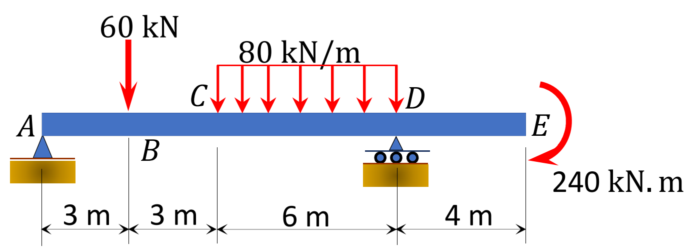


Fig. P7.18. Beam. $E = 200\ GPa$
$I = 600 \times 10^6$ mm^4

Chapter 8

Deflections of Structures: Work-Energy Methods

8.1 Virtual Work Method

The virtual work method, also referred to as the method of virtual force or unit-load method, uses the law of conservation of energy to obtain the deflection and slope at a point in a structure. This method was developed in 1717 by John Bernoulli. To illustrate the principle of virtual work, consider the deformable body shown in Figure 8.1. First, applying a virtual or fictitious unit load $P_v = 1$ at a point Q, where the deflection parallel to the applied load is desired, will create an internal virtual or imaginary load f and will cause point Q to displace by a certain small amount. Then, placing the real external loads P_1, P_2, and M on the same body will cause an internal deformation, dS, and an external deflection of point Q to Q' by an amount Δ.

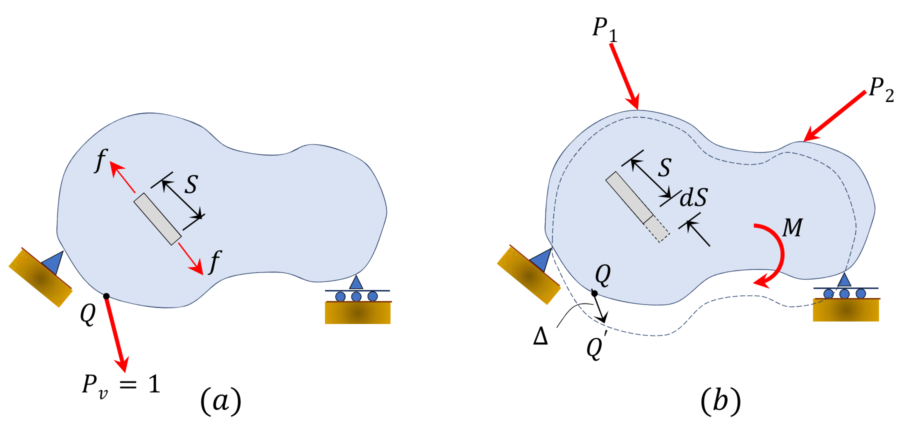


Fig. 8.1. Deformable body.

Upon placement of the real load, the point of application of the virtual load also displaces by Δ, and the applied unit load performs work by traveling the distance Δ. The work done by the virtual forces are as follows:

External work done by the unit load P_v

$$= P_v \times \text{Displacement}$$
$$= 1 \times \Delta \quad (8.1)$$

Internal work done by the virtual load f

$$= f \times dS \quad (8.2)$$

Applying the principle of conservation of energy by equating equation 8.1 and equation 8.2 suggests the following:

External work done = Internal work done

Virtual Loads

$$1 \times \Delta = f \times dS \tag{8.3}$$

Real displacements

where

$P_v = 1$ = external virtual unit load.
f = internal virtual load.
Δ = external displacement caused by real loads.
dS = internal deformation caused by real loads.

Similarly, to obtain the slope at a point on a structure, apply a unit virtual moment M_v at the specified point where the slope is desired, and apply the following equation derived via the principle of conservation of energy:

Virtual Loads

$$1 \times \theta = f_\theta \times dS \tag{8.4}$$

Real displacements

where

$M_v = 1$ = external virtual unit moment.
f = internal virtual load.
θ = external rotational displacement caused by real loads.
dS = internal deformation caused by real loads.

8.1.1 Virtual Work Formulation for the Deflection and Slope of Beams and Frames

To develop the equations for the computation of deflection of beams and frames using the virtual work principles, consider the beam loaded as shown in Figure 8.2a. The deflection at point C due to the applied external loads is required. First, removing the loads P and W and applying a virtual unit load $P_v = 1$ will cause elementary forces and deformations to develop in the bar, and a small deflection to occur at C, as follows:

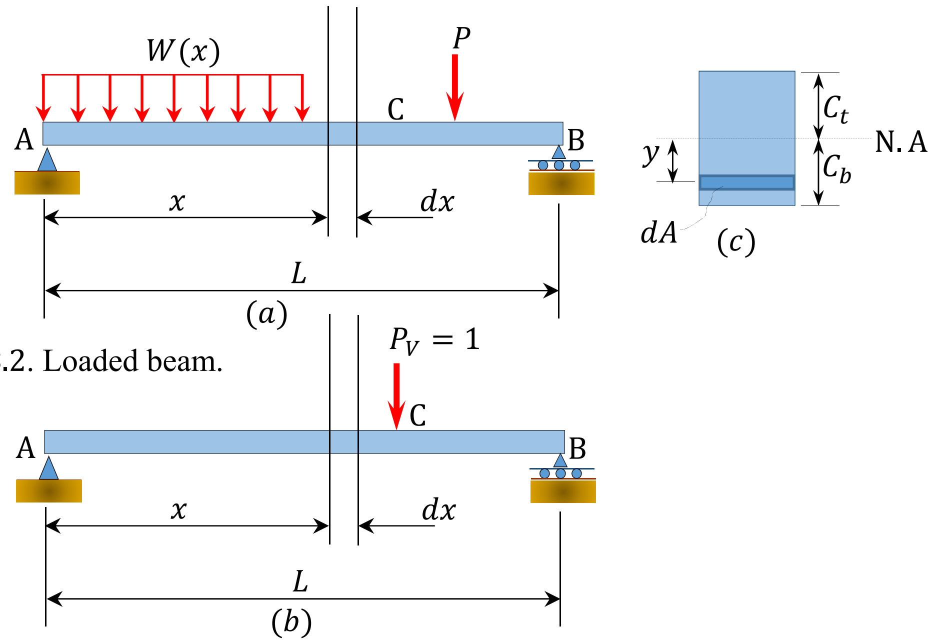


Fig. 8.2. Loaded beam.

The stress acting on the differential cross-sectional area dA at a distance x from the left-end support due to a virtual unit load is as follows:

$$\sigma' = \frac{my}{I} \tag{8.5}$$

where

m = internal virtual moment at the section at a distance xe from the left-end support due to the virtual unit load.
I = moment of inertia of the section.

The force acting on the differential area due to the virtual unit load is written as follows:

$$f = \sigma' dA = \left(\frac{my}{I}\right) dA \tag{8.6}$$

The stress due to the external loads *P1* and *P2* on the beam is written as follows:

$$\sigma = \frac{My}{I} \tag{8.7}$$

The deformation of a differential beam length dx at a distance x from the left-end support is as follows:

$$\delta = \varepsilon dx = \left(\frac{\sigma}{E}\right) dx = \left(\frac{My}{EI}\right) dx \tag{8.8}$$

The work done by the force f acting on the differential area due to the deformation of the differential beam length dx is as follows:

$$
\begin{aligned}
dW &= f\delta = \left(\tfrac{my}{I}\right)dA \times \left(\tfrac{My}{EI}\right)dx \\
&= \left(\tfrac{Mmy^2}{EI^2}\right)dAdx
\end{aligned} \tag{8.9}
$$

The internal work done by the total force in the entire cross-sectional area of the beam due to the applied virtual unit load when the differential length of the beam dx deforms by δ can be obtained by integrating with respect to dA, as follows:

$$
\begin{aligned}
\int_A dW &= \left[\int_{A_1}^{A_n}\left(\tfrac{Mmy^2}{EI}\right)dA\right]dx \\
W_i &= \left(\tfrac{Mm}{EI^2}\int y^2 dA\right)dx \\
&= \left[\left(\tfrac{Mm}{EI^2}\right)I\right]dx \\
&= \left(\tfrac{Mm}{EI}\right)dx
\end{aligned} \tag{8.10}
$$

The internal work done W_i in the entire length of the beam due to the applied virtual unit load can now be obtained by integrating with respect to dx, which is written as follows:

$$
W_i = \int_0^L \left(\tfrac{Mm}{EI}\right)dx \tag{8.11}
$$

The external work done W_e by the virtual unit load due to the deflection Δ at point C of the beam caused by the external loads is as follows:

$$
W_e = 1 \times \Delta \tag{8.12}
$$

The principle of conservation of energy is applied to obtain the expression for the computation of the deflection at any point in a beam or frame, which is written as follows:

$$
W_e = W_i
$$

$$
1 \times \Delta = \int_0^L \left(\tfrac{Mm}{EI}\right)dx
$$

$$
\Delta = \int_0^L \left(\frac{Mm}{EI}\right)dx \tag{8.13}
$$

where

1 = external virtual or imaginary unit load on the beam or frame in the direction of the required deflection Δ.

Δ = external displacement at the specified point on a beam or frame caused by the real loads.

M = internal moment in the beam or frame caused by the real load, expressed in terms of the horizontal distance x.

m = internal virtual moment in the beam or frame caused by the external virtual unit load, expressed with respect to the horizontal distance x.

E = modulus of elasticity of the material of the beam or frame.
I = moment of inertia of the cross-sectional area of the beam or frame about its neutral axis.

Similarly, the following expression can be obtained for the computation of the slope at a point in a beam or frame:

$$\theta = \int_0^L \frac{Mm_\theta}{EI} dx \tag{8.14}$$

where

θ = slope or tangent rotation at a point on a beam or frame.
m_θ = internal virtual moment in the beam or frame, expressed with respect to the horizontal distance x, caused by the external virtual unit moment applied at the point where the rotation is required.

Procedure for Determination of Deflection in Beams and Frames by the Virtual Work Method

- Determine the support reactions in the real system using the equations of static equilibrium.
- Write an expression for the moment in the real structure as a function of the horizontal distance x. The number of the equations will depend on the number of regions of the beam due to discontinuous loading.
- Create a virtual system by removing all the loads acting on the beam and applying a unit load or a unit moment at the point where the deflection or slope is desired.
- Write the moment expression for the virtual system in terms of the distance x.
- Substitute the moment expressions into equation 8.1 and integrate to obtain the value of deflection or slope at the point considered.

8.1.2 Virtual Work Formulation for the Deflection of Trusses

Consider the truss shown in Figure 8.3 for the development of the virtual work expression for the determination of the deflection of trusses. The truss is subjected to the loads P_1, P_2, and P_3, and the vertical deflection Δ at joint F is desired. First, remove the loads P_1, P_2, and P_3, and apply a vertical virtual unit load $P_v = 1$ at joint F, as shown in Figure 8.3b. The virtual unit load will cause the virtual internal axial load n_i to act on each member of the truss. Applying the forces P_1, P_2, and P_3 will cause the deflection Δ at joint F and the internal deformation δL_i in each member of the truss.

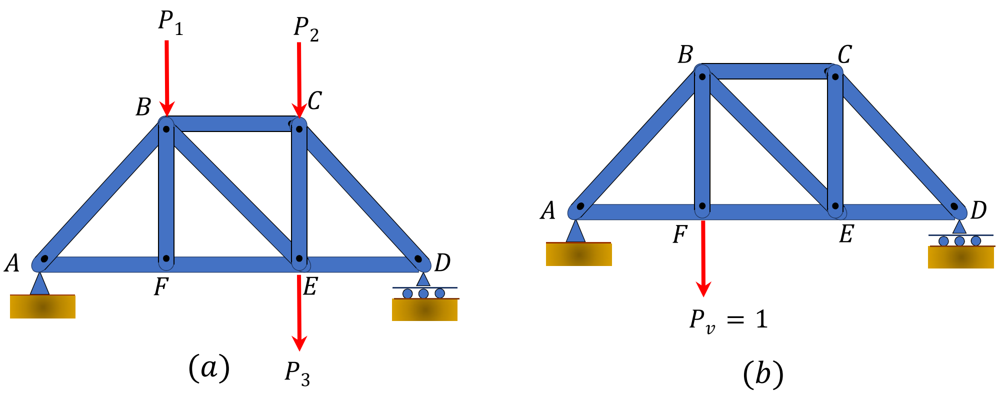


Fig. 8.3. Sample truss.

Using the law of conservation of energy, the work by the virtual unit load at joint F and the virtual internal axial loads on the members of the truss can be written as follows:

External work = internal work

$$1 \times \Delta = \sum_{i=1}^{n} n_i(\delta L_i) \tag{8.15}$$

But, for a member with length L_i, area A_i, and material Young's modulus E_i, the deformation is written as follows:

$$\delta L_i = \frac{N_i L_i}{A_i E_i} \tag{8.16}$$

Thus, the virtual work expression for the deflection of a truss can be written as follows:

$$\Delta = \sum_{i=1}^{n} n_i \left(\frac{N_i L_i}{A_i E_i}\right) \tag{8.17}$$

where

1 = external vertical virtual unit load applied at joint F.
n = internal axial virtual force in each truss member due to the virtual unit load, $P_v = 1$.
N = axial force in each truss member due to the real loads P_1, P_2, and P_3.
Δ = external joint displacement caused by the real loads.
δL = deformation of each truss member caused by the real loads.

Procedure for Determination of Deflection in Trusses by the Virtual Work Method

- Determine the support reactions in the real system with the applied loads using the equations of equilibrium.
- Determine the internal forces *N* in truss members caused by the external loads on the real system.
- Remove all the external loads on the real system and apply a virtual unit load on the joint in the truss in the direction of required deflection.
- Determine the internal virtual forces *n* in the members of the truss caused by the external virtual unit load placed in the joint where the deflection is desired.
- Calculate the deflection Δ in the joint of the truss caused by the real loads using equation 8.17.

Example 8.1

Using the virtual work method, determine the deflection and the slope at a point *B* of the cantilever beam shown in Figure 8.4a. $E = 29 \times 10^3$ksi, $I = 600$ in^4.

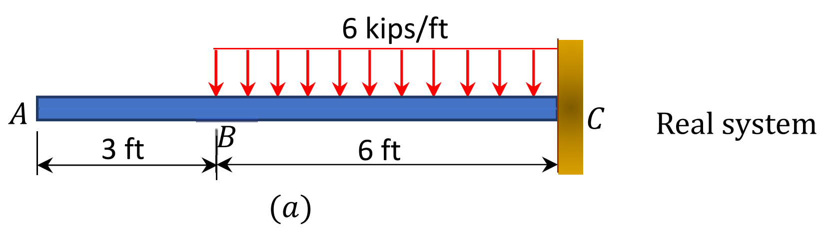
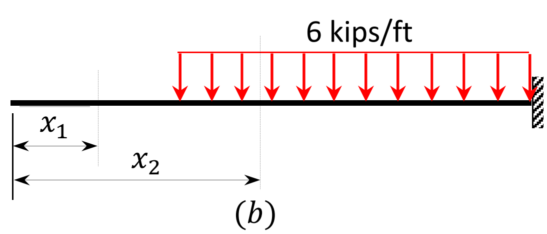
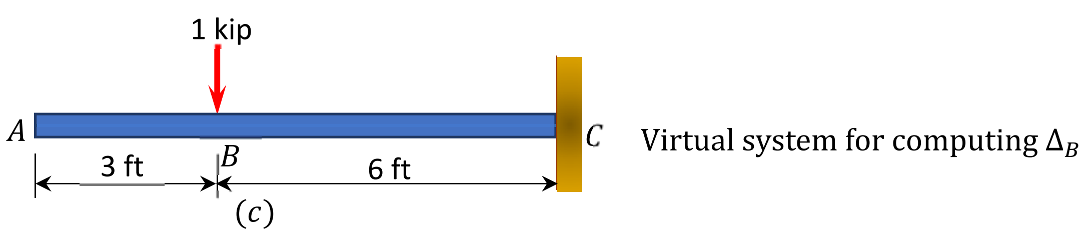
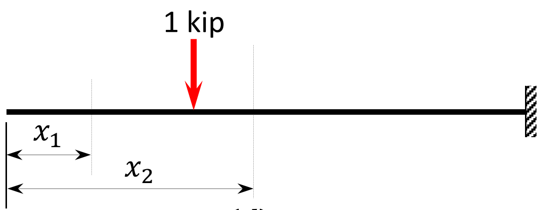


Fig. 8.4. Cantilever beam.







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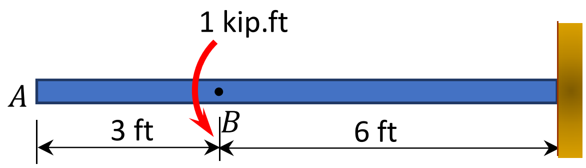
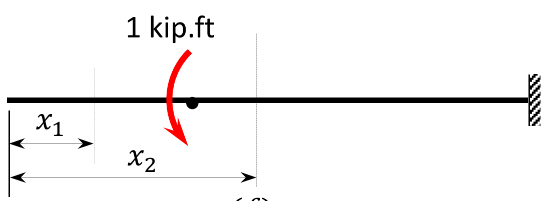


Virtual system for computing θ_B

(e)



(f)

Solution

Real and virtual systems. The real and virtual systems are shown in Figure 8.4a, Figure 8.4c, and Figure 8.4e, respectively. Notice that the real system consists of the external loading carried by the beam, as specified in the problem. The virtual system consists of a unit 1-k load applied at B, where the deflection is required, and 1-k-ft moment applied at the same point where the slope is determined. The bending moments at each portion of the beam, with respect to the horizontal axis, are presented in Table 8.1. Notice that the origin of the horizontal distance, x, for both the real and virtual system is at the free end, as shown in Figure 8.4b, Figure 8.4d, and Figure 8.4f.

Table 8.1. Bending moments at portions of the beam.

Portion	$X-$ Coordinate		Deflection		Slope	
	Origin	Limit (ft)	M	M	M	m_θ
AB	A	0-3	0	0	0	0
BC	A	3-9	$-3(x-3)^2$	$1(x-3)$	$-3(x-3)^2$	-1

Deflection at B. The deflection at the free end of the beam is determined by using equation 8.1, as follows:

$$1\text{ kip}.\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^3 \frac{(0)(0)dx}{EI} + \int_3^9 \frac{-3(x-3)^2(x-3)dx}{EI}$$

$$1\text{ kip}.\Delta_B = \frac{-972\text{ k . ft}^3}{EI}$$

Therefore,

$$\Delta_B = \frac{-972\text{ k . ft}^3\,(12)^3\text{in}^3/\text{ft}^3}{(29\times10^3\text{k/in}^2)(600\text{in}^4)}$$

$$= -0.097\text{ in.} \qquad \Delta_B = 0.097\text{ in.}\uparrow$$

Slope at B. The slope at the free end of the beam is determined by using equation 8.2, as follows:

$$(1\text{ kN. m}).\theta_B = \int_0^L \frac{\text{m}_\theta M}{EI}dx = \int_0^3 \frac{(0)(0)dx}{EI} + \int_3^9 \frac{-3(x-3)^2(-1)dx}{EI}$$

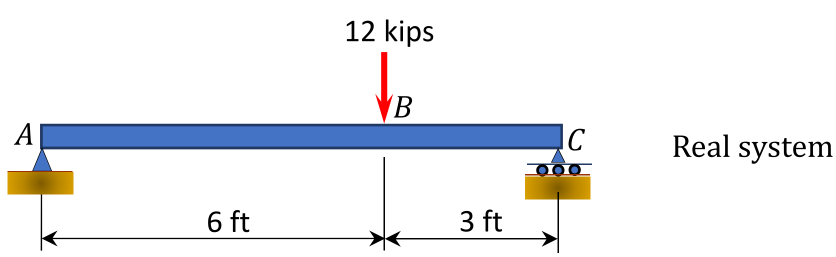
$$(1\text{ k. ft}).\theta_B = \frac{216\text{ k}^2.\text{ ft}^3}{EI} = \frac{216\text{ k}^2.\text{ ft}^3}{(29\times10^3\text{k/in}^2)(600\text{in}^4)}$$

Therefore,

$$\theta_B = \frac{216\text{k. ft}^2}{(29\times10^3\text{k/in}^2)(600\text{in}^4)} = \frac{216(12)^2}{(29\times10^3\text{k/in}^2)(600\text{in}^4)} = 0.0018\text{ rad} \qquad \theta_B = 0.0018\text{ rad}$$

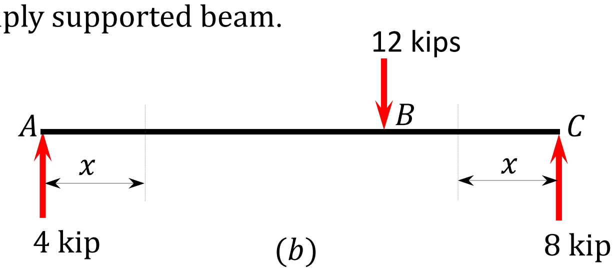
Example 8.2

Using the virtual work method, determine the deflection at B and the slope at C for the simply supported beam subjected to a concentrated load, as shown in Figure 8.5a . EI= constant. $E = 29\times10^3$ksi. $I = 24\text{ in}^4$.



(a)

Fig. 8.5. Simply supported beam.



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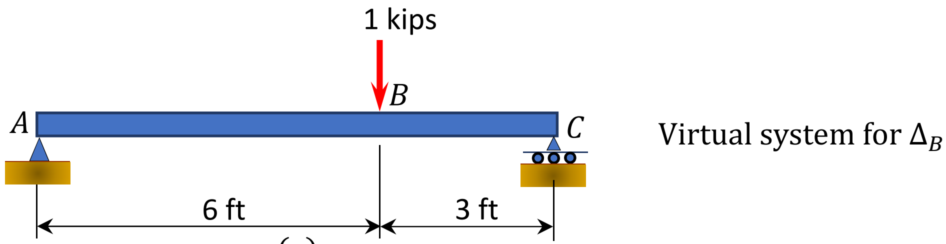

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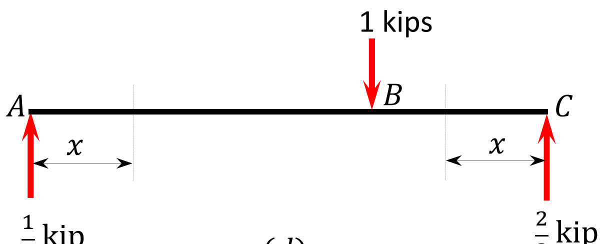

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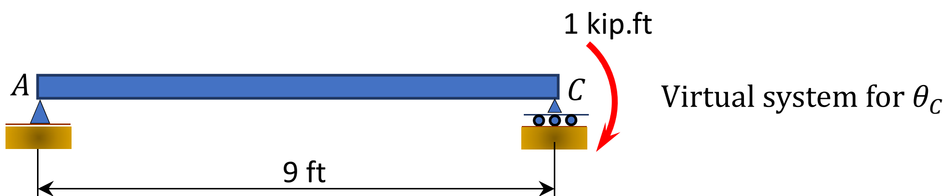

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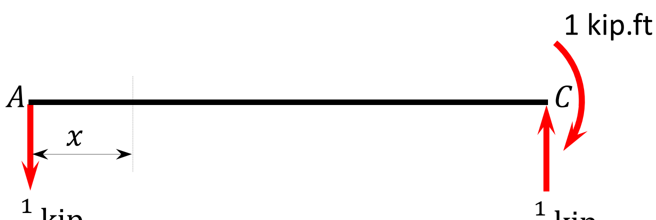


(f)

Solution

Real and virtual systems. The real and virtual systems are shown in Figure 8.5a, Figure 8.5c, and Figure 8.5e, respectively. The bending moments at each portion of the beam, with respect to the horizontal axis, are presented in Table 8.2. The origin of the horizontal distances for both the real and virtual system are shown in Figure 8.5b, Figure 8.5d, and Figure 8.5f.

Table 8.2. Bending moments at portions of the beam.

	x Coordinate		Deflection		Slope	
Portion	Origin	Limits (ft)	M	m	M	m_θ
AB	A	0-6	$4x$	$\frac{x}{3}$	$4x$	$-\frac{x}{9}$
CB	C	0-3	$8x$	$\frac{2x}{3}$	$8x$	$\frac{x}{9}-1$

Deflection at *B*. The deflection at *B* can be determined by using equation 8.1, as follows:

$$1\text{ kip}.\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^6 \frac{(4x)\left(\frac{x}{3}\right)dx}{EI} + \int_0^3 \frac{(8x)\left(\frac{2x}{3}\right)dx}{EI}$$

$$1\text{ kip}.\Delta_B = \frac{144\text{ k}.\text{ ft}^3}{EI}$$

Therefore,

$$\Delta_B = \frac{144\text{ k}.\text{ft}^3\,(12)^3\text{in}^3/\text{ft}^3}{(29\times10^3\text{ k/in}^2)(24\text{ in}^4)} = 0.36\text{ in} \qquad \Delta_B = 0.36\text{ in.}\downarrow$$

The positive value indicates deflection in the direction of the applied virtual load.

Slope at *C*. The slope at *C* can be determined by using equation 8.2, as follows:

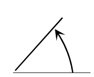
$$(1\text{ k}.\text{ft}).\theta_C = \int_0^L \frac{m_\theta M}{EI}dx = \int_0^6 \frac{(4x)\left(-\frac{x}{9}\right)dx}{EI} + \int_0^3 \frac{(8x)\left(\frac{x}{9}-1\right)dx}{EI}$$

$$(1\text{ k}.\text{ft}).\theta_C = -\frac{60\text{ k}^2.\text{ ft}^3}{EI} = -\frac{60\text{ k}^2.\text{ ft}^3}{(29\times10^3\text{k/in}^2)(24\text{in}^4)}$$

Therefore,

$$\theta_C = -\frac{60\text{k}.\text{ ft}^2}{\left(29\times\frac{10^3\text{k}}{\text{in}^2}\right)(24\text{ in}^4)} = -\frac{60(12)^2}{(29\times10^3\text{k/in}^2)(24\text{ in}^4)}$$

$$= -0.012\text{ rad} \qquad \theta_C = 0.012$$



Example 8.3

Using the virtual work method, determine the deflection at *B* and the slope at *D* for the compound beam shown in Figure 8.6a. $E = 200$ GPa and $I = 250\times10^6\text{ mm}^4$.

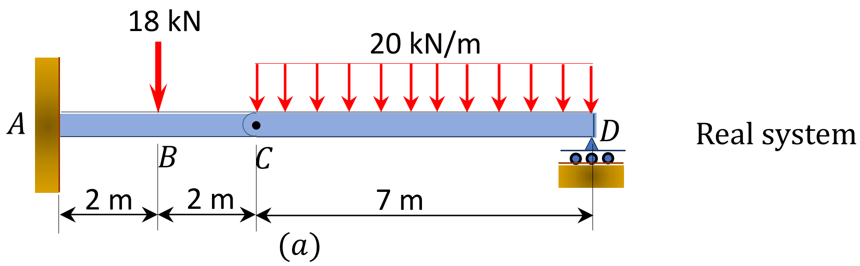
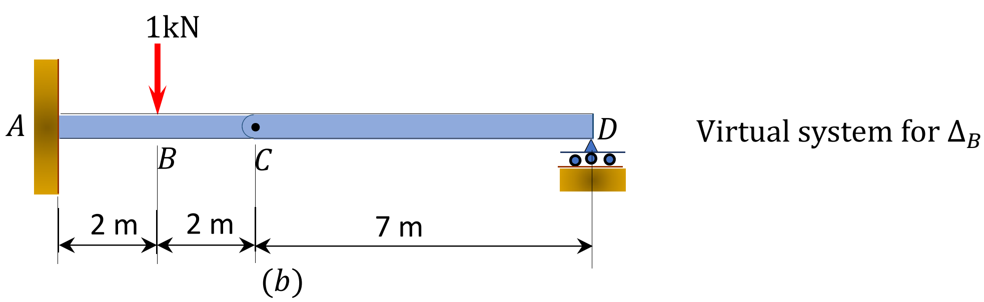
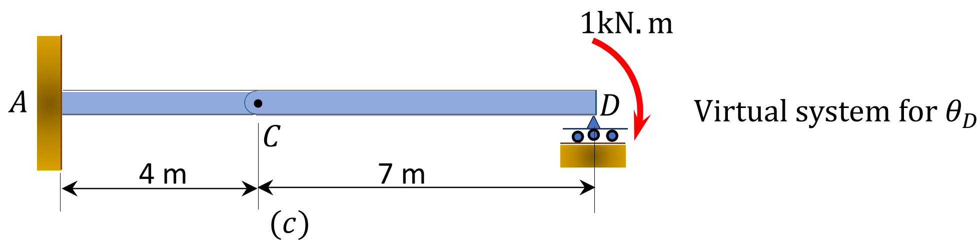


Fig. 8.6. Compound beam.





Solution

Real and virtual systems. The real and virtual systems are shown in Figure 8.6a, Figure 8.6b, and Figure 8.6c, respectively. The bending moment at each portion of the beam, with respect to the horizontal axis, are presented in Table 8.3.

Table 8.3. Bending moments at portions of the beam.

Portion	X − Coordinate		Deflection		Slope	
	Origin	Limit (ft)	M	m	M	m_θ
DC	D	0-7	$70x - 10x^2$	0	$70x - 10x^2$	$\frac{x}{7} - 1$
CB	C	0-2	$-70x$	0	$-70x$	$\frac{x}{7}$
BA	C	2-4	$-70x - 18(x-2)$	$-x$	$-70x - 18(x-2)$	$\frac{x}{7}$

Deflection at B. The deflection at B can be determined using equation 8.1, as follows:

$$1\text{ kN}.\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^7 \frac{(0)(70x-10x^2)dx}{EI} + \int_0^2 \frac{(0)(-70x)dx}{EI} + \int_2^4 \frac{(-x)[-70x-18(x-2)]dx}{EI}$$

$$1\text{ kN}.\Delta_B = \frac{1426.67\text{ kN}^2.\text{m}^3}{EI}$$

Therefore,

$$\Delta_B = \frac{1426.67\text{ kN.m}^3}{(200\times10^6\text{ kN/m}^2)(250\times10^6\text{mm}^4)(10^{-12}\text{m}^4/\text{mm}^4)} = 0.0285\text{ m} \qquad \Delta_B = 28.5\text{ mm}\downarrow$$

Slope at D. The slope at D can be determined using equation 8.2, as follows:

$$(1\text{ kN.m}).\theta_D = \int_0^L \frac{m_\theta M}{EI}dx = \int_0^7 \frac{\left(\frac{x}{7}-1\right)(70x-10x^2)dx}{EI} + \int_0^2 \frac{\left(\frac{x}{7}\right)(-70x)dx}{EI} + \int_2^4 \frac{\left(\frac{x}{7}\right)[-70x-18(x-2)]dx}{EI}$$

$$(1\text{ kN.m}).\theta_D = -\frac{516.31\text{ kN}^2.\text{m}^3}{EI}$$

Therefore,

$$\theta_D = \frac{-516.31\text{ kN.m}^3}{(200\times10^6\text{ kN/m}^2)(250\times10^6\text{mm}^4)(10^{-12}\text{m}^4/\text{mm}^4)} = -0.0103\text{ rad} \qquad \theta_D = 0.0103\text{ rad}$$

The negative sign indicates that the rotation at point D is in the direction opposite to the applied virtual moment.

Example 8.4

Using the virtual work method, determine the slope at joint A of the frame shown in Figure 8.7a. $E = 29\times10^3$ksi and $EI = 700\text{ in}^4$.

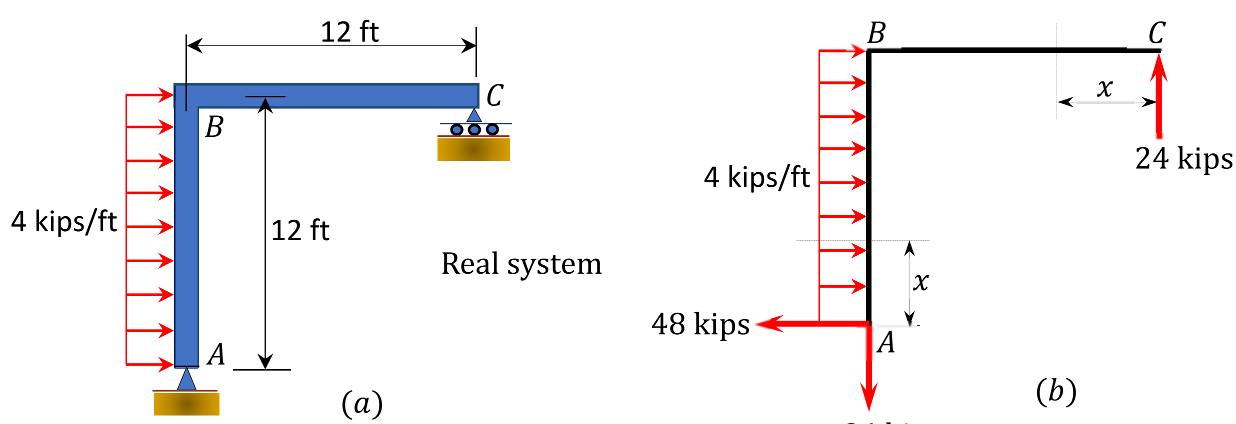
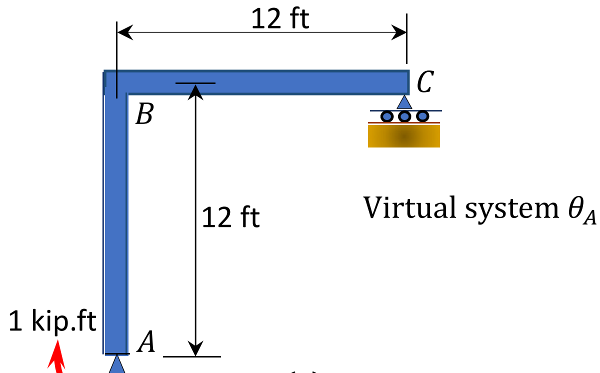


Fig. 8.7. Frame.



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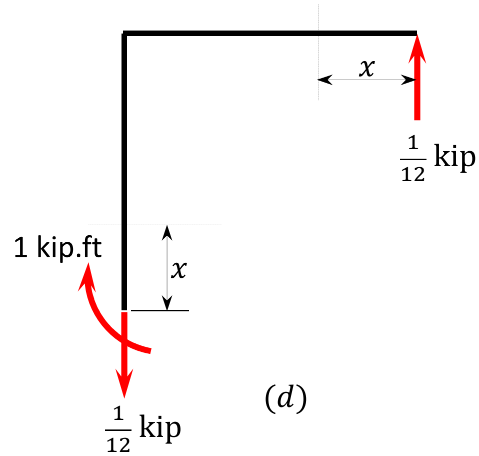


(d)

Solution

Real and virtual systems. The real and virtual systems are shown in Figure 8.7a and Figure 8.7c, respectively. The bending moment at each segment of the beam and column of the frame are presented in Table 8.4, and their origins are shown in Figure 8.7b and Figure 8.7d.

Table 8.4. Bending moments at portions of the beam.				
	$X-$ Coordinate		Deflection	
Portion	Origin	Limit	M	m
AB	0	0-12	$48x-2x^2$	1
CB	0	0-12	$24x$	$\frac{x}{12}$

Slope at A. The slope at A can be determined by using equation 8.2, as follows:

$$(1\ \text{k.ft}).\theta_A = \int_0^L \frac{m_\theta M}{EI}dx = \int_0^{12} \frac{(1)(48x-2x^2)dx}{EI} + \int_0^{12} \frac{(24x)(\frac{x}{12})dx}{EI}$$

$$(1\ \text{k.ft}).\theta_A = \frac{3456\ \text{k}^2.\text{ft}^3}{EI}$$

Therefore,

$$\theta_A = \frac{3456\text{k.ft}^2}{(29\times10^3\text{k/in}^2)(700\ \text{in}^4)} = \frac{3456(12)^2}{(29\times10^3\text{k/in}^2)(700\ \text{in}^4)} = 0.0245\ \text{rad} \qquad \theta_A = 0.0245\ \text{rad}$$

Example 8.5

Using the virtual work method, determine the vertical deflection at A of the frame shown in Figure 8.8a. $E = 200$ GPa and $I = 250 \times 10^6$ mm^4.

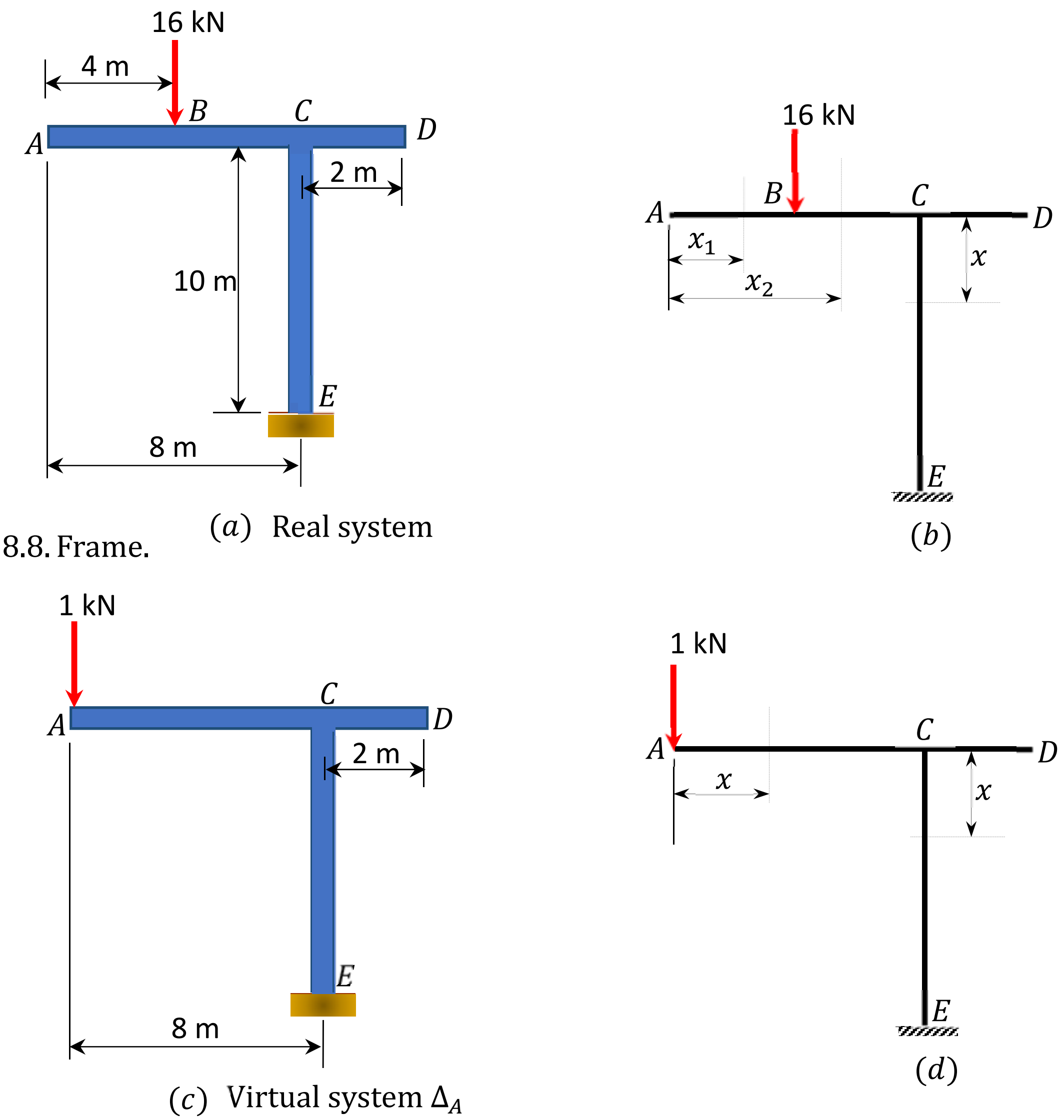


Fig. 8.8. Frame.

Solution

Real and virtual systems. The real and virtual systems are shown in Figure 8.8a and Figure 8.8c, respectively. The bending moment at each segment of the beam and column of the frame are presented in Table 8.5, and their origins are shown in Figure 8.8b and Figure 8.8d.

Table 8.5. Bending moments at portions of the beam.				
Portion	$X-$ Coordinate		Deflection	
	Origin	Limit	M	m
AB	A	0-4	0	$-x$
BC	A	4-8	$-16(x-4)$	$-x$
CE	C	0-10	-64	-8

Deflection at *A*. The deflection at *A* can be determined by using equation 8.1, as follows:

$$1\text{ kN}.\Delta_A = \int_0^L \frac{mM}{EI}dx = \int_0^4 \frac{(0)(-x)dx}{EI} + \int_4^8 \frac{(-16(x-4))(-x)dx}{EI} + \int_0^{10} \frac{(-8)(-64)dx}{EI}$$

$$1\text{ kN}.\Delta_A = \frac{853.33\text{ kN}^2.\text{m}^3}{EI}$$

Therefore,

$$\Delta_A = \frac{853.33\text{ kN.m}^3}{(200\times10^6\text{ kN/m}^2)(250\times10^6\text{mm}^4)(10^{-12}\text{m}^4/\text{mm}^4)} = 0.017\text{ m} \qquad \Delta_A = 17\text{ mm}\downarrow$$

Example 8.6

Using the virtual work method, determine the horizontal deflection at joint *B* of the truss shown in Figure 8.9a. $E = 12000$ ksi and $A = 3$ in^2.

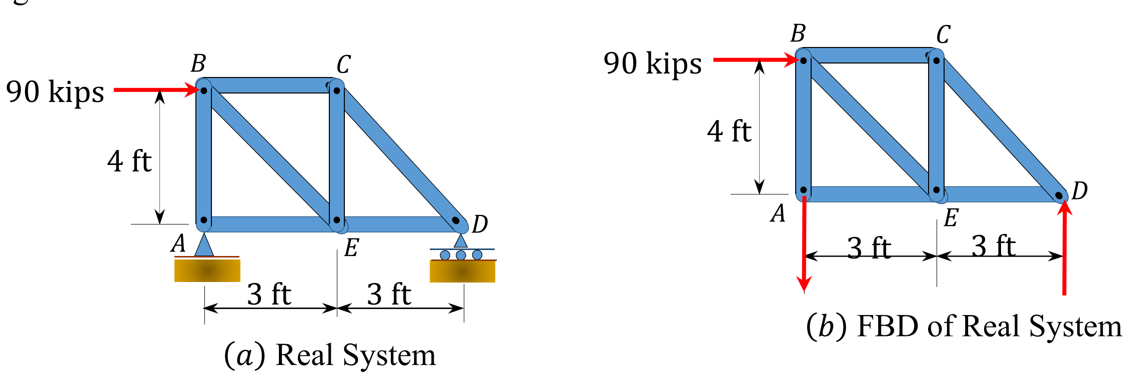
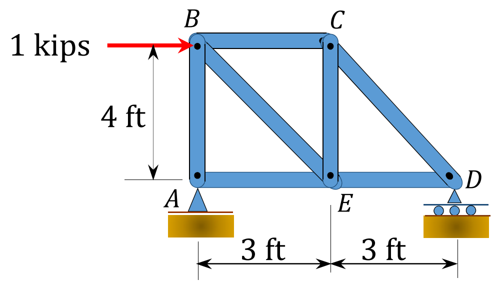
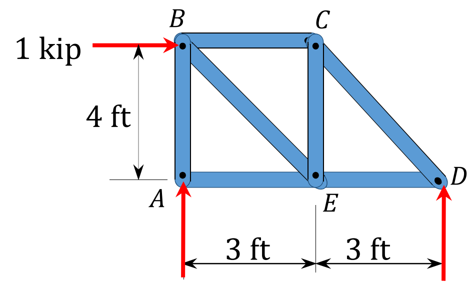


Fig. 8.9. Truss.



(c) Virtual System



(d) FBD of Virtual System

Solution

Real and virtual systems. The real and virtual systems are shown in Figure 8.9. Notice that the real system consists of the external loading carried by the truss, as specified in the problem. The virtual system consists of a unit 1-k load applied at B, where the deflection is desired, and a 1-k-ft moment applied also at B, where the slope is required.

Truss analysis. The analysis of the real system used to obtain the forces in members is presented below. The forces in members in the virtual system are obtained by dividing the forces in the real system by the applied external load, as the deflection is desired for the same joint where the deflection is required.

Support reactions. The reactions are computed by the application of the equations of equilibrium, as follows:

$$+\circlearrowleft \sum M_D = 0$$

$$6A_y - 90(4) = 0$$

$$A_y = 60 \text{ kips} \qquad\qquad A_y = 60 \text{ kips} \downarrow$$

$$+\rightarrow \sum F_x = 0$$

$$-A_x + 90 = 0$$

$$A_x = 90 \text{ kips} \qquad\qquad A_x = 90 \text{ kips} \leftarrow$$

$$+\uparrow \sum F_y = 0$$

$$D_y - 60 = 0$$

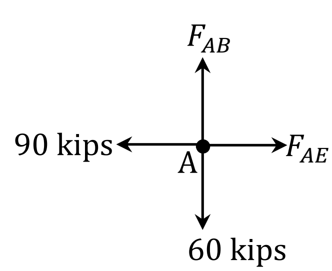
$$D_y = 60 \text{ kN} \qquad\qquad D_y = 60 \text{ kips} \uparrow$$

Joint A.

$$+\uparrow \sum F_y = 0$$

$$F_{AB} - 60 = 0$$

$$F_{AB} = 60 \text{ kips}$$



$+\rightarrow \sum F_x = 0$

$F_{AE} - 90 = 0$

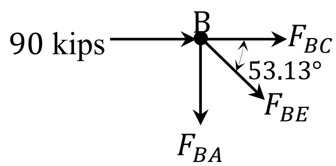
$F_{AE} = 90$ kips

Joint *B*.

$+\uparrow \sum F_y = 0$

$-F_{BE}\sin 53.13° - F_{BA} = 0$

$F_{BE} = -\frac{F_{BA}}{\sin 53.13°} = -\frac{60}{\sin 53.13°} = -75$ kips



$+\rightarrow \sum F_x = 0$

$F_{BE}\cos 53.13° + 90 + F_{BC} = 0$

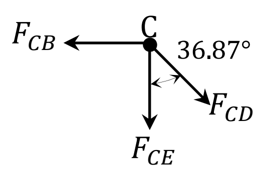
$F_{BC} = -F_{BE}\cos 53.13° - 90 = -(-75)\cos 53.13° - 90 = -45$ kips

Joint *C*.

$+\rightarrow \sum F_x = 0$

$F_{CD}\sin 36.87° - F_{CB} = 0$

$F_{CD} = \frac{F_{CB}}{\sin 36.87°} = -\frac{45}{\sin 36.87} = -75$ kips

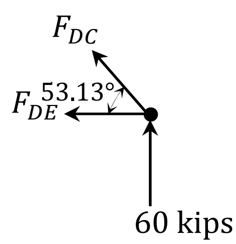


$+\uparrow \sum F_y = 0$

$-F_{CE} - F_{CD}\cos 36.87° = 0$

$F_{CE} = -F_{CD}\cos 36.87° = -(-75)\cos 36.87° = 60$ kips

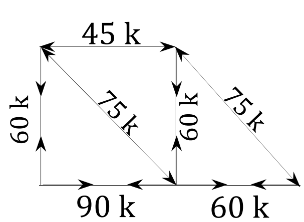
Joint *D*.



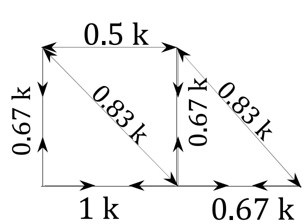
$+\rightarrow \sum F_x = 0$

$-F_{DC}\cos 36.87° - F_{DE} = 0$

$F_{DE} = -F_{DC}\cos 36.87° = -(-75)\cos 36.87° = 60$ kips



Real System



Virtual System

Horizontal deflection at B. The desired horizontal deflection at joint B is computed using equation 8.17, as presented in Table 8.6.

Table 8.6. Horizontal deflections.				
Member	Length (ft)	N (kip)	N (kip)	NnL (k^2.ft)
AB	4	60	0.67	160.8
AE	3	90	1	270
BC	3	45	0.5	67.5
BE	5	75	0.83	311.25
CD	5	75	0.83	311.25
CE	4	60	0.67	160.8
DE	3	60	0.67	120.6
				$\sum NnL$ =1401.4

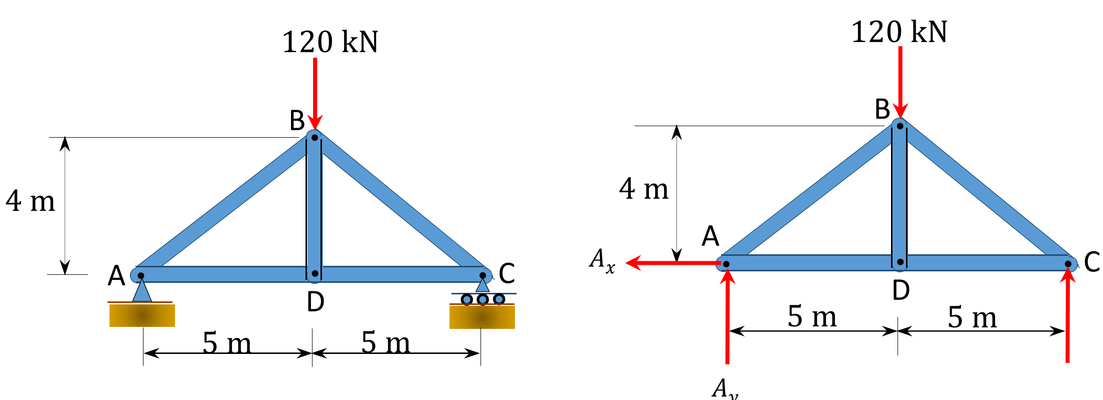
$$1(\Delta_B) = \frac{1}{EA}\sum NnL$$

$$(1\text{ k})\Delta_B = \frac{1401.4}{12000(12^2)(3)(12^{-2})}$$

$$\Delta_B = 0.039\text{ ft} = 0.47\text{ in} \qquad \Delta_B = 0.47\text{ in} \downarrow$$

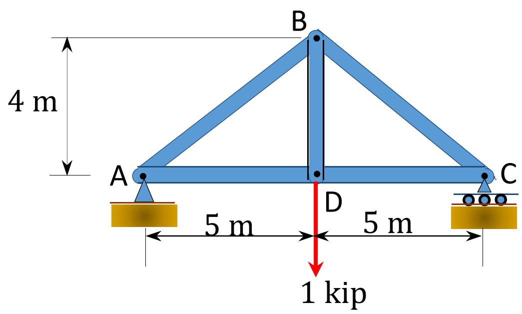
Example 8.7

Using the virtual work method, determine the vertical deflection at joint D of the truss shown in Figure 8.10a. $E = 200$ GPa and $A = 5\text{ cm}^2$.

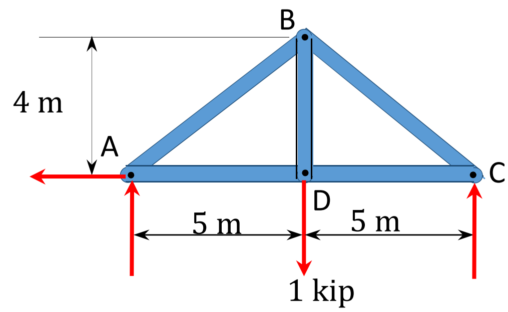


(a) Real System (b) FBD of Real System

Fig. 8.10. Truss.



(c) Virtual System



(d) FBD of Virtual System

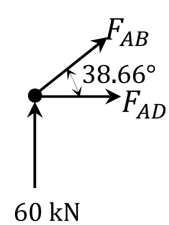
Solution

Real and virtual systems. The real and virtual systems are shown in Figure 8.10. Notice that the real system consists of the external loading carried by the beam, as specified in the problem. The reactions in both supports in the real system are the same by reason of symmetry in loading and equal 60 kN. The virtual system consists of a unit 1-k load applied at *B*, where the deflection is required, and a 1-k-ft moment applied at the same point, where the slope is to be determined. The bending moment at each portion of the beam with respect to the horizontal axis is presented in Table 8.7. Notice that the origin of the horizontal distance *x* for both the real and virtual system is at the free end, as shown in Figure 8.10.

Real system-truss analysis.

Joint *A*.

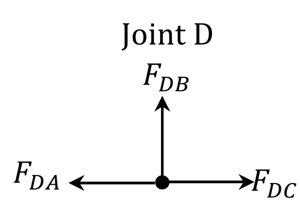
$$+\uparrow \sum F_y = 0$$
$$F_{AB} sin38.66° + 60 = 0$$
$$F_{AB} = -96.05 \text{ kN}$$

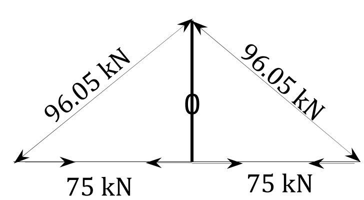


$$+\rightarrow \sum F_x = 0$$
$$F_{AB} cos38.66° + F_{AD} = 0$$
$$F_{AD} = -F_{AB} cos38.66°$$
$$= -(-96.05) cos38.66° = 75 \text{ kN}$$

Joint *D*.

$$+\uparrow \sum F_y = 0$$
$$F_{DB} = 0$$
$$+\rightarrow \sum F_x = 0$$
$$-F_{DA} + F_{DC} = 0$$
$$F_{DC} = F_{DA} = 75 \text{ kN}$$





(e) Real System Axial Forces, N

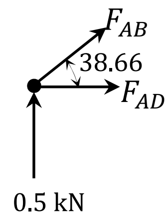
Virtual system truss analysis.

Joint *A*.

$+\uparrow \sum F_y = 0$

$F_{AB} sin38.66° + 0.5 = 0$

$F_{AB} = -0.08$ kN

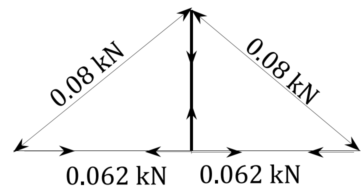


$+\rightarrow \sum F_x = 0$

$F_{AB} cos38.66° + F_{AD} = 0$

$F_{AD} = -F_{AB} cos38.66°$

$= -(-0.08) cos38.66° = 0.062$ kN



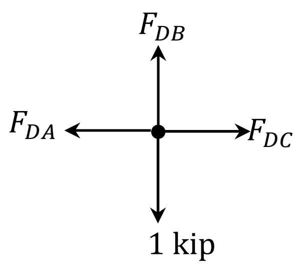
(f) Virtual System- Axial Forces, n

Joint *D*.

$+\uparrow \sum F_y = 0$

$F_{DB} - 1 = 0$

$F_{DB} = 1$kN



$+\rightarrow \sum F_x = 0$

$-F_{DA} + F_{DC} = 0$

$F_{DC} = F_{DA} = 0.062$ kN

Vertical deflection at *D*. The desired vertical deflection at joint *D* is calculated using equation 8.17, as presented in Table 8.7.

Table 8.7. Vertical deflections.				
Member	Length (m)	N	n	NnL ($kN^2.m$)
AB	6.4	-96.05	-0.08	49.18
AD	5	75	0.062	23.25
BC	6.4	-96.05	-0.08	49.18
BD	4	0	1	0
DC	5	75	0.062	23.25
				$\sum NnL = 144.86$

$$1(\Delta_D) = \frac{1}{EA}\sum NnL$$

$$(1\ kN)\Delta_D = \frac{144.86}{200(10^6)(0.0005)} kN.m$$

$$\Delta_D = 1.45 \times 10^{-3} m = 1.45\ mm \qquad \Delta_D = 1.45\ mm \downarrow$$

8.2 Energy Methods

The energy method for the determination of deflection is based on Alberto Castigliano's second theorem, which was published in 1879. The theorem states the following:

The deflection or rotation in a specified direction and at a specified point in a linear elastic, statically determinate structure subjected to a given force or couple is equal to the partial derivative of the total external work or the total internal energy, with respect to the applied force or couple in the direction of the force or couple.

Castigliano's second theorem, with respect to the applied force, can be expressed mathematically, as follows:

$$\Delta = \frac{\partial W}{\partial P} = \frac{\partial U}{\partial P} \tag{8.18}$$

where

Δ = deflection at the point of application of the load P in the direction of the load P.
W = work done.
U = strain energy.

8.2.1 Energy Method Formulation for Beams and Frames

Equation 8.18 can be mathematically manipulated to include moment and is written as follows:

$$\Delta = \frac{\partial U}{\partial P} = \frac{\partial U}{\partial M} \times \frac{\partial M}{\partial P} \tag{8.19}$$

The total internal work done or strain energy stored in a beam or frame due to gradually applied real loads can be expressed as follows:

$$W = U = \int \frac{M^2}{2EI} dx \tag{8.20}$$

The partial derivative of equation 8.20, with respect to the moment, is as follows:

$$\frac{\partial U}{\partial M} = \int \left(\frac{2M}{2EI}\right) dx = \int \left(\frac{M}{EI}\right) dx \tag{8.21}$$

Substituting equation 8.21 into equation 8.19 yields the following equation for the computation of deflection for beams and frames by the energy method:

$$\Delta = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx \tag{8.22}$$

With respect to the applied couple, Castigliano's second theorem can be expressed mathematically as follows:

$$\theta = \frac{\partial W}{\partial M'} = \frac{\partial U}{\partial M'} \tag{8.23}$$

where

θ = rotation at the point of application and direction of the couple M'.

Equation 8.23 can be mathematically manipulated to include the moment, as follows:

$$\theta = \frac{\partial U}{\partial M'} = \frac{\partial U}{\partial M} \times \frac{\partial M}{\partial M'} \tag{8.24}$$

Substituting equation 8.21 into equation 8.24 suggests the following equation for the computation of slopes for beams and frames by the energy method:

$$\theta = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M'}\right) dx \tag{8.25}$$

Example 8.8

Using Castigliano's second theorem, determine the deflection and the slope at the free end of the cantilever beam shown in Figure 8.11a.

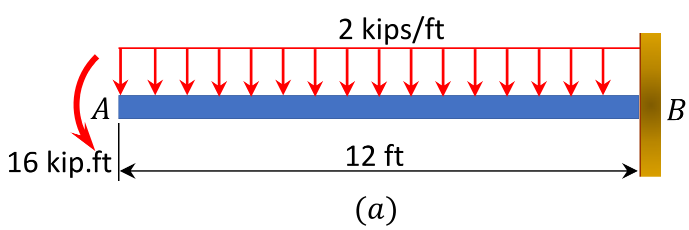
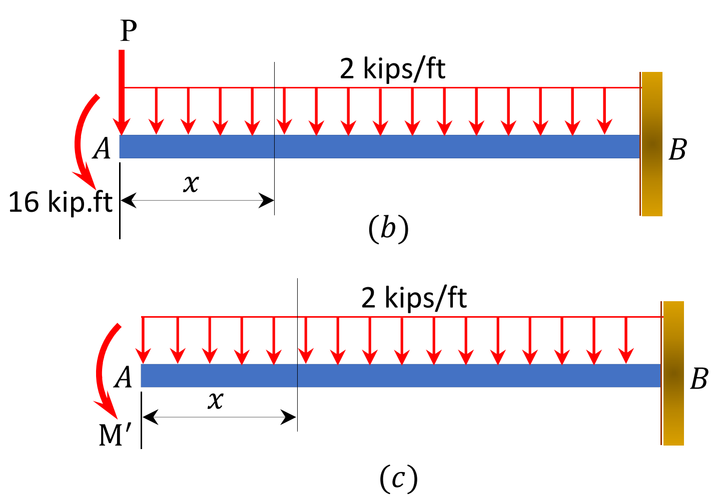


Fig. 8.11. Cantilever beam.



Solution

Placement of imaginary force P and couple M'. The force P and the moment M' are placed at point A, where the deflection and slope are desired, as shown in Figure 8.11b and Figure 8.11c, respectively.

Bending moment. To determine the deflection, write the bending moment equation for the beam as a function of the force P. To determine the slope, write the bending moment equation for the beam as a function of M'. The x coordinates for the moment equations are also shown in Figure 8.11b and Figure 8.11c. Compute the partial derivatives $\frac{\partial M}{\partial P}$ and $\frac{\partial M}{\partial M'}$, and then apply Castigliano's equation 8.22 and equation 8.25 to determine the deflection and slope.

Deflection at A.

$$M = -16 - Px - x^2$$

$$\frac{\partial M}{\partial P} = -x$$

Setting $P = 0$ and applying Castigliano's theorem suggests the following:

$$\Delta_A = \int_0^L \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right)dx$$

$$\Delta_A = \int_0^{12} \left(\frac{-16-x^2}{EI}\right)(-x)dx$$

$$= \frac{6336\ \text{k. ft}^3}{EI} = \frac{6336(12)^3\ \text{k. ft}^3}{(29000)(1500)} = 0.252\ \text{in} \qquad \Delta_A = 0.252\ \text{in} \downarrow$$

Slope at A.

$$M = -M' - x^2$$

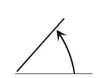
$$\frac{\partial M}{\partial M'} = -1$$

Setting $M' = 16\ \text{k.ft}$ and applying Castigliano's theorem suggests the following:

$$\theta_A = \int_0^L \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M'}\right)dx$$

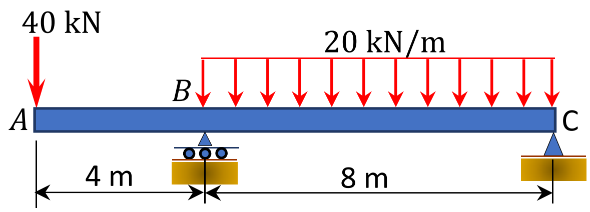
$$\theta_A = s\int_0^{12} \left(\frac{-16-x^2}{EI}\right)(-1)dx$$

$$= \frac{768\ \text{k. ft}^2}{EI} = \frac{768(12)^2}{(29000)(1500)} = 0.0025\ \text{rad} \qquad \theta_A = 0.0025\ \text{rad}$$



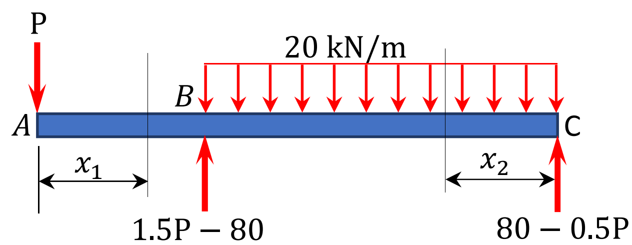
Example 8.9

Using Castigliano's second theorem, determine the deflection at point A of the beam with the overhang shown in Figure 8.12a.



(a)

Fig. 8.12. Beam with overhang.



(b)

Solution

Placement of imaginary force P. The force P is placed at point A, where the deflection is desired, as shown in Figure 8.12b. The x coordinates for the moment equations are also shown in this figure.

Bending moment. Compute the support reactions and write the bending moment equations for segments AB and BC of the beam as a function of the force P. The x coordinates for the moment equations are also shown in Figure 8.12b. Compute the partial derivatives $\frac{\partial M}{\partial P}$, and then apply Castigliano's equation 8.22 to compute the deflection.

Segment AB. $(0 < x_1 < 4)$

$$M_1 = -Px_1$$

$$\frac{\partial M_1}{\partial P} = -x_1$$

Segment BC. $(0 < x_2 < 8)$

$$M_2 = (80 - 0.5P)x_2 - 10x_2^2$$

$$\frac{\partial M_2}{\partial P} = -0.5x_2$$

Setting $P = 40$ kN and applying Castigliano's theorem suggests the following:

$$\Delta_A = \int_0^L \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right)dx$$

$$\Delta_A = \int_0^4 \left(\frac{-40x_1}{EI}\right)(-x_1)dx + \int_0^8 \left(\frac{100x_2 - 10x_2^2}{EI}\right)(-0.5x_2)dx$$

$$= \frac{-2560 \text{ kN . m}^3}{EI} = \frac{-2560 \text{ kN. m}^3}{(200\times10^6 \text{ kN/m}^2)(800\times10^6 \text{mm}^4)(10^{-12}\text{m}^4/\text{mm}^4)} = -0.016 \text{ m}$$

$$\Delta_A = 16 \text{ mm} \uparrow$$

Example 8.10

Using Castigliano's second theorem, determine the rotation of joint C of the frame shown in Figure 8.13a.

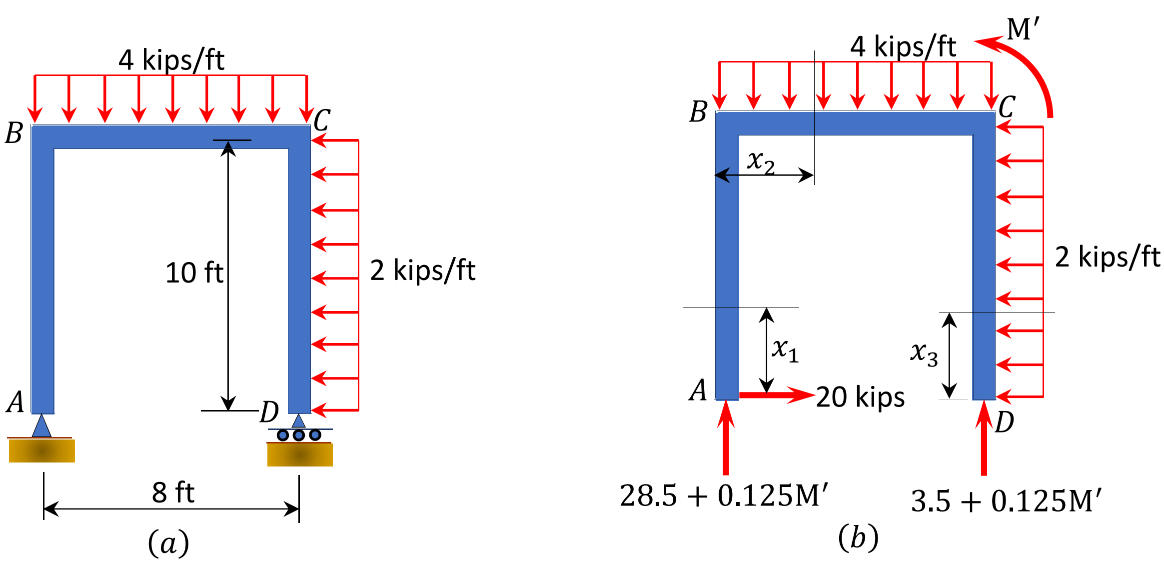


Fig. 8.13. Frame.

Solution

Placement of imaginary couple M'. The couple force M' is placed at point C, where the rotation is desired, as shown in Figure 8.13b.

Bending moment. Compute the support reactions and write the bending moment equations for the columns AB and DC and the beam BC of the frame as a function of the couple M'. Compute the partial derivatives $\frac{\partial M}{\partial M'}$, and then apply Castigliano's equation 8.25 for the computation of rotation.

Column AB. $(0 < x_1 < 10)$

$M_1 = -20x_1$

$\frac{\partial M_1}{\partial M'} = 0$

Beam BC. $(0 < x_2 < 8)$

$M_2 = (28.5 + 0.125\mathrm{M}')x_2 - 2x_2^2$

$\frac{\partial M_2}{\partial M'} = 0.125x_2$

Column DC. $(0 < x_3 < 10)$

$M_3 = -x_3^2$

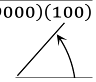
$\frac{\partial M_3}{\partial M'} = 0$

Setting $M' = 16$ k . ft and applying Castigliano's theorem suggests the following:

$$\theta_A = \int_0^L \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M'}\right)dx$$

$$\theta_A = \int_0^{10}\left(\frac{-20x_1}{EI}\right)(0)dx + \int_0^{8}\left(\frac{28.5x_2 - 2x_2^2}{EI}\right)(0.125x_2)dx + \int_0^{10}\left(\frac{-x_3^2}{EI}\right)(0)dx$$

$$= \frac{351.57\ \mathrm{k.\ ft^2}}{EI} = \frac{351.57(12)^2}{(29000)(100)} = 0.017\ \mathrm{rad}$$

$$\theta_A = 0.017\ \mathrm{rad}$$ 

8.2.2 Energy Method Formulation for Trusses

Equation 8.18 can be mathematically manipulated to include axial force, as follows:

$$\Delta = \frac{\partial U}{\partial P} = \frac{\partial U}{\partial N} \times \frac{\partial N}{\partial P} \tag{8.26}$$

The total internal work done or strain energy stored in members of a truss due to gradually applied external loads is as follows:

$$W = U = \sum \frac{N^2 L}{2AE} \tag{8.27}$$

The partial derivative of equation 8.27, with respect to the axial load, is as follows:

$$\frac{\partial U}{\partial N} = \sum \frac{2NL}{2AE} = \sum \frac{NL}{AE} \tag{8.28}$$

To determine the deflection at any joint of a truss, use the energy method by substituting equation 8.28 into equation 8.26 to obtain the following equation:

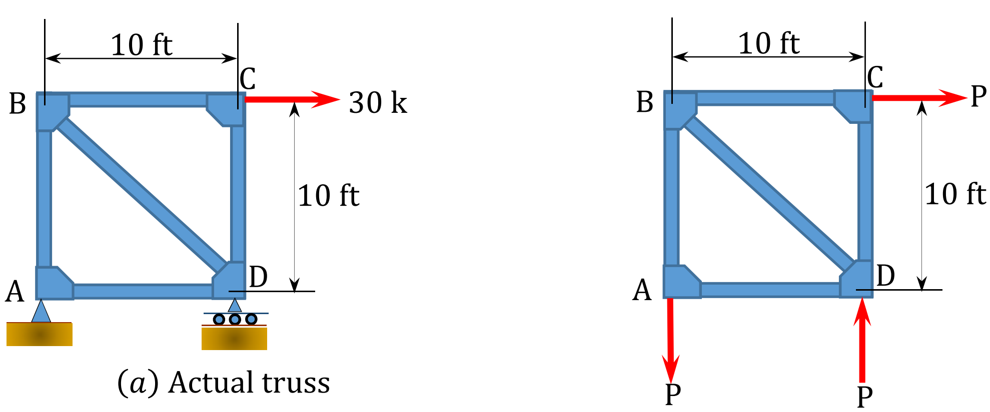
$$\Delta = \sum \left(\frac{NL}{AE}\right)\left(\frac{\partial N}{\partial P}\right) \tag{8.29}$$

where

N = internal axial force in each member due to external load.
$\frac{\partial N}{\partial P}$ = axial force in each member due to unit load applied at the joint and in the direction of the required deflection.
L = length of member.
A = area of a member.
E = modulus of elasticity of a member.

Example 8.11

Using Castigliano's second theorem, determine the horizontal deflection at joint C of the truss shown in Figure 8.14a.



(a) Actual truss

(b) Truss for computing Δ_c

Fig. 8.14. Truss.

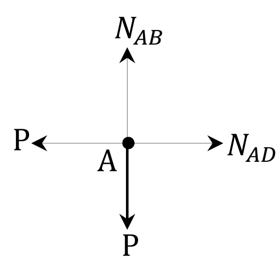
Solution

Placement of imaginary force P. The force P is placed as a replacement for the 30k force at point C, where the horizontal deflection is desired, as shown in Figure 8.14b.

Member axial forces. Compute the support reactions and obtain the member-axial forces in terms of the imaginary force P. Member-axial forces are determined by using the method of joint, as shown below. To find the horizontal deflection at C, compute the partial derivatives $\frac{\partial M}{\partial P}$ and apply Castigliano's equation 8.22. Member lengths, axial forces, and partial derivatives with respect to the fictitious force P are shown in Table 8.8.

Analysis of truss (fig. 8.14b).

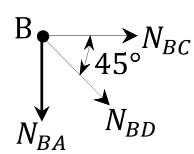
Joint A.



$$+\uparrow \sum F_y = 0$$
$$N_{AB} - P = 0$$
$$N_{AB} = P$$

$$+\rightarrow \sum F_x = 0$$
$$N_{AD} - P = 0$$
$$N_{AD} = P$$

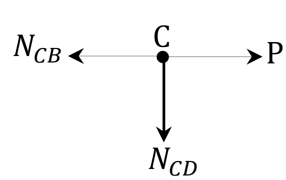
Joint B.



$$+\uparrow \sum F_y = 0$$
$$-N_{BA} - N_{BD}\cos 45° = 0$$
$$N_{BD} = -\frac{N_{BA}}{\cos 45°} = -\frac{P}{\cos 45°} = -1.41P$$

$$+\rightarrow \sum F_x = 0$$
$$N_{BC} + N_{BD}\cos 45° = 0$$
$$N_{BC} = -N_{BD}\cos 45° = -(-1.41P)\cos 45° = P$$

Joint C.



$$+\uparrow \sum F_y = 0$$

$N_{CD} = 0$

$$+\rightarrow \sum F_x = 0$$
$$-N_{CB} + P = 0$$
$$N_{CB} = P$$

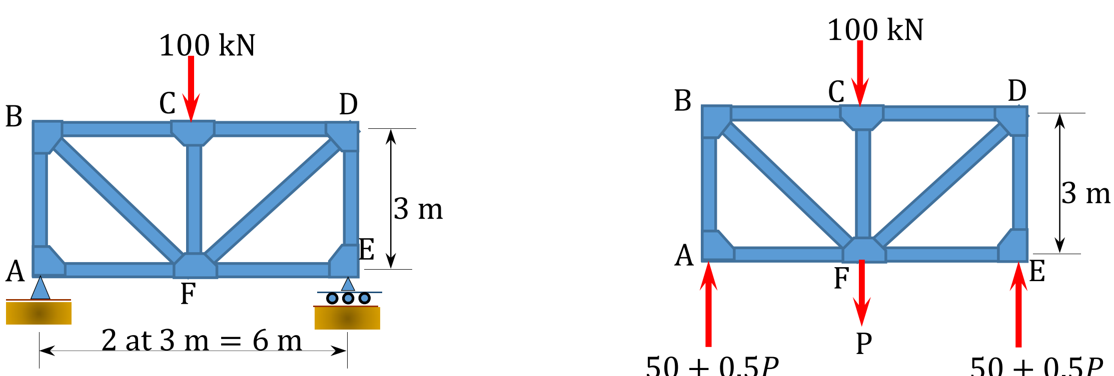
Table 8.8. Member lengths, axial forces, and partial derivatives with respect to the fictitious force P.

Member	L(ft) (ft)	N(kip) (kip)	$\frac{\partial N}{\partial P}$ (kip/kip)	$N(P = 30k)$ (kip)	$N(\partial N/\partial P)L$ (kip. ft)
AB	10	P	1	30	300
AD	10	P	1	30	300
BC	10	P	1	30	300
BD	14.14	-1.14P	-1.14	-34.2	551.29
CD	10	0	0	0	0
					$\sum N\left(\frac{\partial N}{\partial P}\right)L = 1451.29$

$$\Delta_c = \frac{1}{EA}\sum N\left(\frac{\partial N}{\partial P}\right)L = \frac{1451.29}{EA}\text{k. ft} = \frac{1451.29(12)}{(29{,}000)(0.6)} = 0.083\text{ft} = 1 \text{ in} \qquad \Delta_c = 1 \text{ in} \rightarrow$$

Example 8.12

Using Castigliano's second theorem, determine the vertical deflection at joint F of the truss shown in Figure 8.15a.. Members have the same cross-sectional area of 600 mm^2 and $E = 200$ GPa.



(a) Actual truss

(b) Truss for computing Δ_F

Fig. 8.15. Truss.

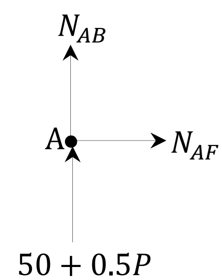
Solution

Placement of imaginary force P. The force P is placed at joint F, where the vertical deflection is desired, as shown in Figure 8.15b.

Member axial forces. Compute support reactions and obtain member-axial forces in terms of the imaginary force P. Member-axial forces are determined by using the method of joint, as shown below. To find the vertical deflection at F, compute the partial derivatives $\frac{\partial M}{\partial P}$ and apply Castigliano's equation 8.22. Member lengths, axial forces, and partial derivatives with respect to the fictitious force P are shown in Table 8.9.

Analysis of truss (fig. 8.15b).

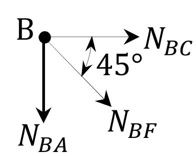
Joint A.



$$+\uparrow \sum F_y = 0$$
$$N_{AB} + 50 + 0.5P = 0$$
$$N_{AB} = -50 - 0.5P$$

$$+\rightarrow \sum F_x = 0$$
$$N_{AF} \ 0$$

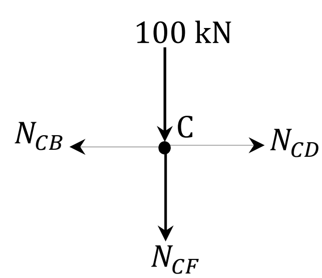
Joint B.



$$+\uparrow \sum F_y = 0$$
$$-N_{BA} - N_{BF}\cos 45° = 0$$
$$N_{BF} = -\frac{N_{BA}}{\cos 45°} = -\frac{(-50-0.5P)}{\cos 45°} = 70.71 + 0.7071P$$

$$+\rightarrow \sum F_x = 0$$
$$N_{BC} + N_{BF}\cos 45° = 0$$
$$N_{BC} = -N_{BF}\cos 45° = -(70.71 + 0.7071P)$$

Joint C.



$$+\uparrow \sum F_y = 0$$
$$-N_{CF} - 100 = 0$$
$$N_{CF} = -100\ kN$$

$$+\rightarrow \sum F_x = 0$$

$$-N_{CB} + N_{CD} = 0$$

$$N_{CD} = N_{CB} = -(70.71 + 0.7071P)$$

Table 8.9. Member lengths, axial forces, and partial derivatives with respect to the fictitious force P.

Member	L (m)	N (kN)	$\frac{\partial N}{\partial P}$ (kN/kN)	$N(P=0)$ (kN)	$N(\partial N/\partial P)L$ (kN.m)
AB	3	$-50-0.5P$	-0.5	-50	75
AF	3	0	0	0	0
BC	3	-70.71-0.7071P	-0.7071	-70.71	150
BF	4.24	$70.71+0.7071P$	0.7071	70.71	212
CF	3	100	0	100	0
CD	3	-70.71-0.7071P	-0.7071	-70.71	150
DF	4.24	$70.71+0.7071P$	0.7071	70.71	212
DE	3	$-50-0.5P$	-0.5	-50	75
EF	3	0	0	0	0
					$\sum N\left(\frac{\partial N}{\partial P}\right)L = 874$

$$\Delta_c = \frac{1}{EA}\sum N\left(\frac{\partial N}{\partial P}\right)L = \frac{874}{EA}\text{kN.m} = \frac{1451.29(12)}{(29{,}000)(0.6)} = 0.083\text{ft} = 1\text{ in} \qquad \Delta_c = 1\text{ in} \rightarrow$$

Chapter Summary

Principle of virtual work: The principle of virtual work states that if a body acted upon by several external forces is in a state of equilibrium and is subjected to a small virtual displacement, the virtual work done by the externally applied forces is zero. This principle can be expressed mathematically, as follows:

$$W_e = W_i$$

The expressions for the determination of deflection by virtual work method for beams and trusses are as follows:

Beams and Frames: $$1(\Delta) = \int_0^L \frac{Mm_v}{EI}dx$$

Trusses: $$1(\Delta) = \sum n_i\left(\frac{N_iL_i}{A_iE_i}\right)$$

Principle of conservation of energy: The principle of conservation of energy states that the work done by external forces acting on an elastic body in equilibrium are equal to the strain energy stored in the body. This principle can be expressed mathematically, as follows:

$$W(or\ U_e) = U_i$$

The energy method for the determination of deflection is based on Alberto Castigliano's second theorem. The theorem states that the deflection in a specified direction and at a specified point in a linear elastic structure subjected to a given force is equal to the partial derivative of the total external work or the total internal energy with respect to the applied force. The expressions for the determination of deflection by Castigliano's second theorem for beams and trusses are as follows:

Beams and Frames: $\Delta = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$

Trusses: $\Delta = \sum \left(\frac{NL}{AE}\right)\left(\frac{\partial N}{\partial P}\right)$

Practice Problems

8.1 Using the virtual work method, determine the slope and deflection at point *A* of the cantilever beams shown in Figure P8.1 and Figure P8.2.

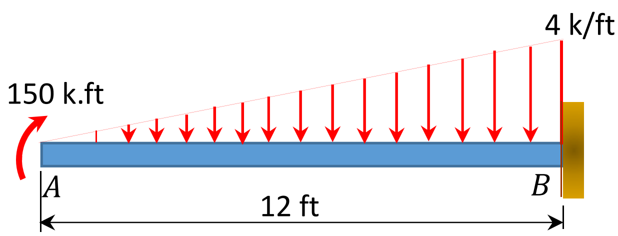


Fig. P8.1. Cantilever beam.

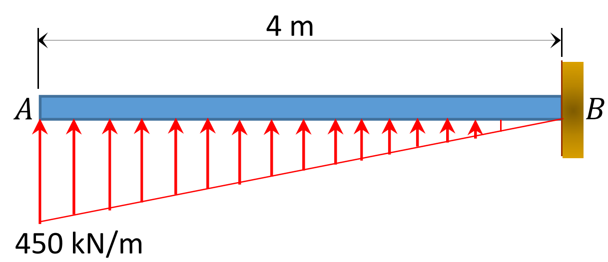


Fig. P8.2. Cantilever beam.

8.2 Determine the deflection at point *D* of the beams shown in Figure P8.3 and Figure P8.4.

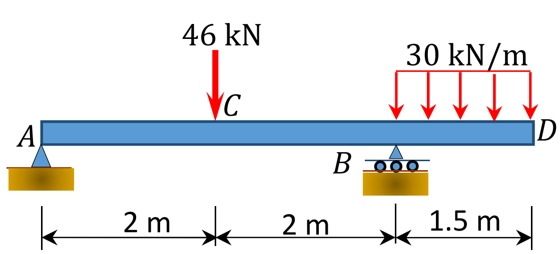


Fig. P8.3. Beam.

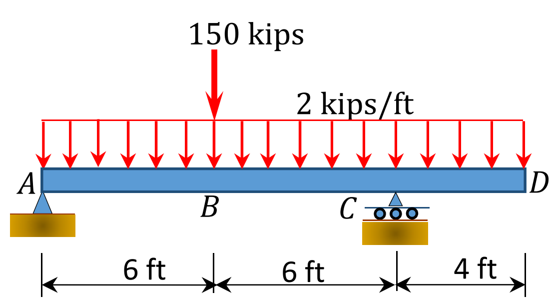


Fig. P8.4. Beam.

8.3 Using the energy method, determine the slope at support *B* of the beams shown in Figure P8.5 and Figure P8.6.

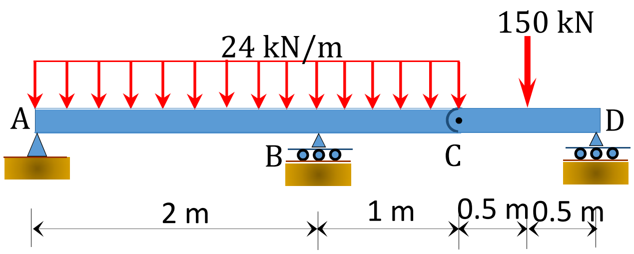


Fig. P8.5. Beam.

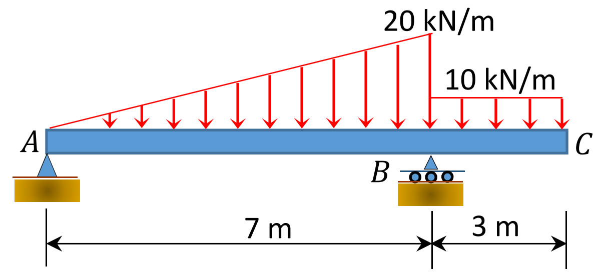


Fig. P8.6. Beam.

8.4 Using the virtual work method, determine the deflection at point *H* of the trusses shown in Figure P8.7 through Figure P8.10.

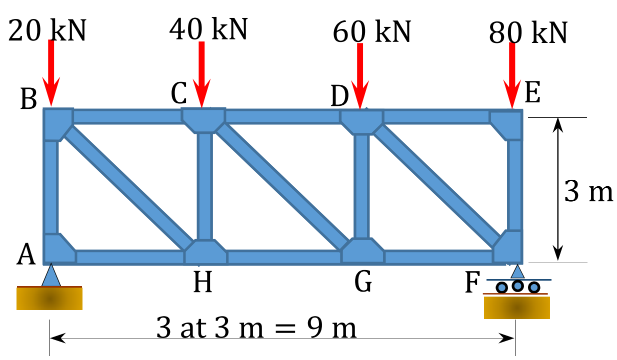


Fig. P8.7. Truss.

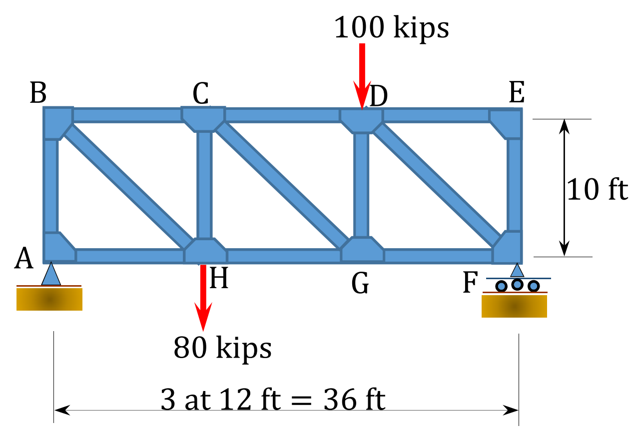


Fig. P8.8. Truss.

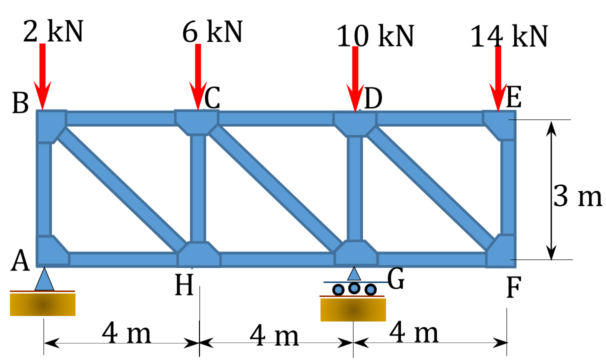


Fig. P8.9. Truss.

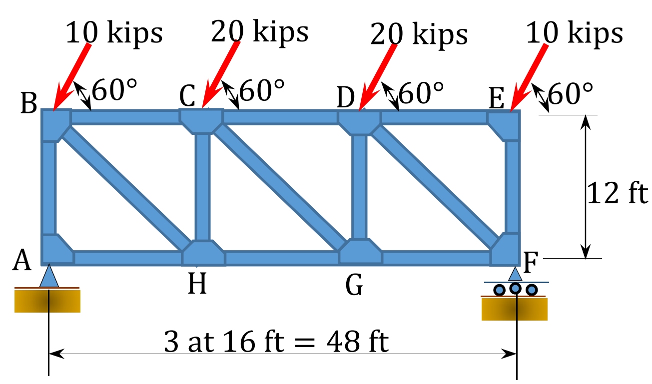


Fig. P8.10. Truss.

8.5 Using the energy method, determine the deflection at point *F* of the trusses shown in Figure P8.11 and Figure P8.12.

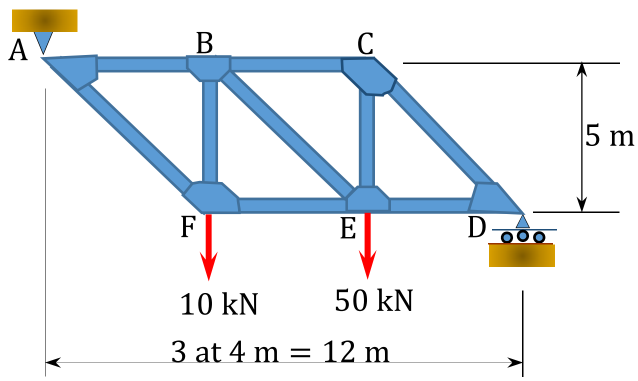


Fig. P8.11. Truss.

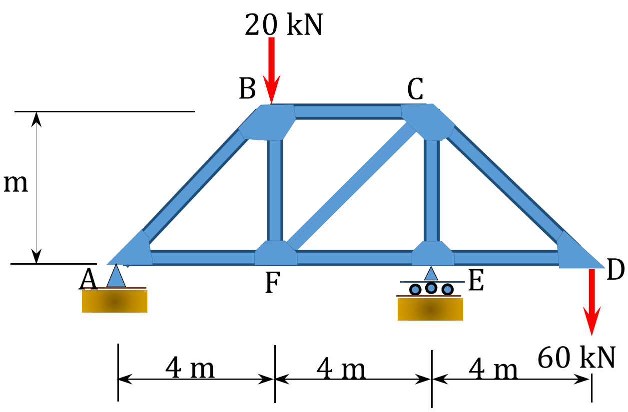


Fig. P8.12. Truss.

8.6 Using the virtual work method, determine the horizontal deflection at joint *C* of the trusses shown in Figure P8.13 and Figure P8.14.

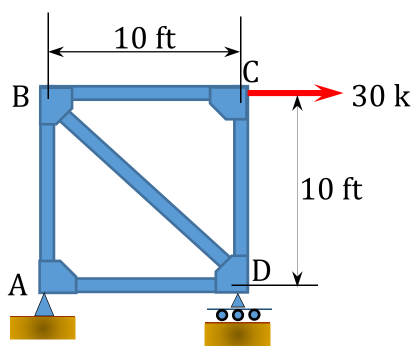


Fig. P8.13. Truss.

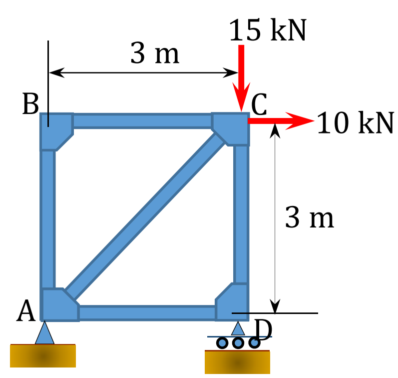


Fig. P8.14. Truss.

Chapter 9

Influence Lines for Statically Determinate Structures

9.1 Introduction

Structures such as bridges and overhead cranes must be designed to resist moving loads as well as their own weight. Since structures are designed for the critical loads that may occur in them, influence lines are used to obtain the position on a structure where a moving load will cause the largest stress. Influence lines can be defined as a graph whose ordinates show the variation of the magnitude of a certain response function of a structure as a unit load traverses across the structure. Response functions of a structure may include axial forces in members, support reactions, bending moments, shear forces, and deflection at specific points in the structure.

It is very important to emphasize the need for students to fully grasp the afore-stated definition, since most of the confusion and difficulty encountered when drawing influence lines stems from a lack of understanding of the difference between this topic and the bending moment and shearing force topics detailed in chapter four. A shearing force or bending moment diagram shows the magnitude of the shearing force or bending moments at different points of the structure due to the static or stationary loads that are acting on the structure, while the influence lines for certain functions of a structure at a specified point of the structure show the magnitude of that function at the specified point when a unit moving load traverses across the structure. The influence lines of determinate structures can be obtained by the static equilibrium method or by the kinematic or Muller-Breslau method. Influence lines by the static equilibrium method are referred to as quantitative influence lines, as they require some calculations, while those by kinematic method are known as the qualitative influence lines, as the method enables the analyzer to obtain the correct shape of the influence lines without any quantitative efforts. In the subsequent sections, students will consider how to construct the influence lines for beams and trusses using these two methods.

9.2 Influence Lines for Statically Determinate Beams by Static Equilibrium Method

To grasp the basic concept of influence lines, consider the simple beam shown in Figure 9.1a. Statics help to determine the magnitude of the reactions at supports *A* and *B*, and the shearing force and bending moment at a section *n*, as a unit load of arbitrary unit, moves from right to left.

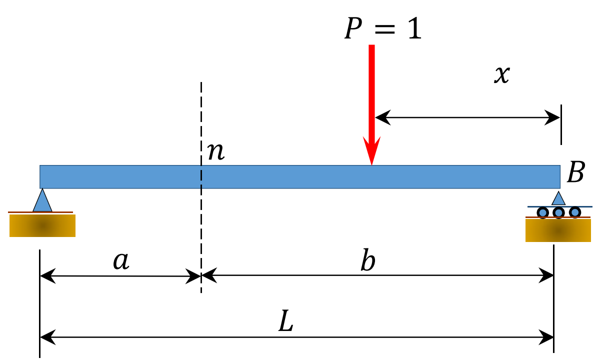


Fig. 9.1a. Simple beam.

9.2.1 Beam Reactions

Taking the moment about B as the unit load moves a distance x from the right-hand end suggests the following:

$$\curvearrowleft + \sum M_B = 0$$
$$-R_A L + Px = 0$$
$$R_A = \frac{Px}{L} \tag{9.1}$$

Setting $P = 1$ suggests the following:

$$R_A = \frac{x}{L} \tag{9.2}$$

Equation 9.2 is the expression for the computation of the influence line for the left-end reaction of a simply supported beam. The influence line for R_A can be represented graphically by putting some values of x into the equation. Since the equation is linear, two points should be enough.

When $x = 0, R_A = 0$
When $x = L, R_A = 1$

The graphical representation of the influence line for R_A is shown in Figure 9.1b, and the ordinate of the diagram corresponding to any value of x gives the magnitude of R_A at that point.

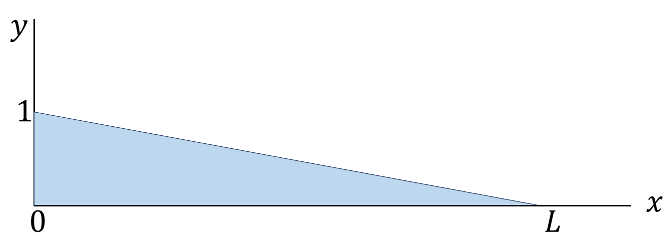


Fig. 9.1b. Influence line for R_A.

Similarly, the expression for the influence line for the reaction R_B is found by taking the moment about A.

$$\sum M_A = 0$$
$$R_B L - P(L - x) = 0$$
$$R_B = \frac{P(L-x)}{L} \tag{9.3}$$

Setting $P = 1$ into equation 9.3 suggests the following:

$$R_B = \frac{(L-x)}{L} \tag{9.4}$$

Equation 9.4 is the expression for the computation of the influence line for the right-end reaction of a simply supported beam. Substituting some values for x into the equation helps to construct the influence line diagram for R_B.

When $x = 0,\ R_B = 1$
When $x = L,\ R_B = 0$

The graphical representation of the influence line for R_B is shown in Figure 9.1c.

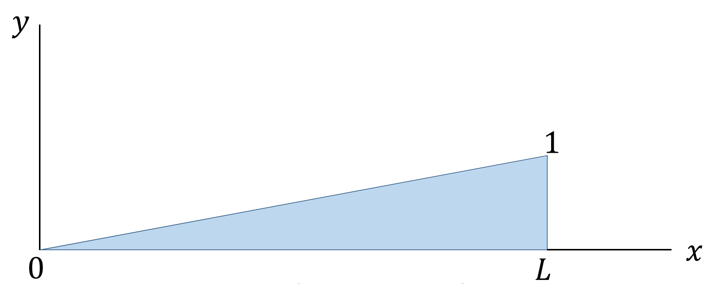


Fig. 9.1c. Influence line for R_B.

9.2.2 Shearing Force at Section n

When the unit load is on the right side of the section, the shear force at the section can be computed considering the transverse forces on the left side of the section, as follows:

Shearing force, $V = R_A = \frac{x}{L}$
When $x = 0, V = 0$
When $x = b, V = \frac{b}{L}$

When the unit load is on the left side of the section, it is easier to compute the shear force in the section by considering the forces on the right side of section, as follows:

$V = -R_B = -\frac{(L-x)}{L}$
When $x = b, V = -\frac{(L-b)}{L} = -\frac{a}{L}$
When $x = L, V = 0$

The graphical representation of the influence line for the shearing force at a section n of the simple beam is shown in Figure 9.1d.

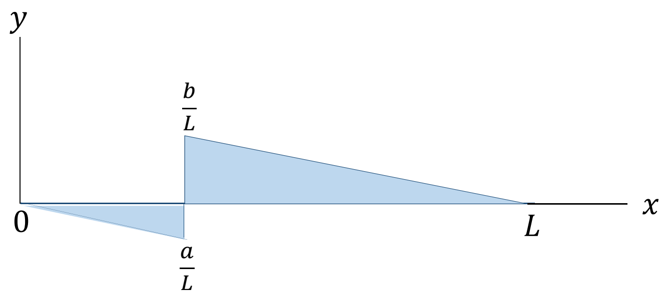


Fig. 9.1d. Influence line for shear at section *n.*

9.2.3 Bending Moment at a Section *n*

When the unit load is on the right side of the section, the bending moment at the section can be computed as follows:

$$M = R_A(L-x) = \frac{x}{L}(L-x)$$
When $x = 0, M = 0$
When $x = b, M = \frac{ab}{L}$

When the unit load is on the left side of section, the bending moment at the section can be computed as follows:

$$M = R_B x = \frac{(L-x)}{L}x$$
When $x = 0, M = 0$
When $x = b, M = \frac{ab}{L}$

The graphical representation of the influence line for the bending moment at a section *n* of the simple beam is shown in Figure 9.1e.

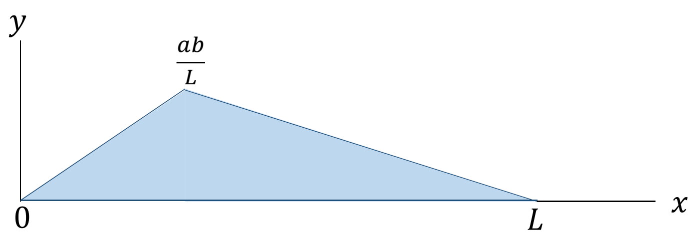


Fig. 9.1e. Influence line for moment at section *n.*

9.3 Construction of Influence Lines

In practice, influence lines are mostly constructed, and the values of the functions are determined by geometry. The procedure for the construction of influence lines for simple beams, compound beams, and trusses will be outlined below and followed by a solved example to clarify the problem. For each case, one example will be solved immediately after the outline.

9.3.1 Simple Beams Supported at Their Ends

The procedures for the construction of the influence lines (I.L.) for some functions of a beam supported at both ends are as follows:

9.3.1.1 Influence Line for Left End Support Reaction, R_A (Fig. 9.2)

(a) At the position of the left end support (point A), along the y-axis, plot a value $+1$ (point A').

(b) Draw a line joining point A' and the zero ordinate at point B. Point B is at the position of support B.

(c) The triangle $AA'B$ is the influence line for the left-end support reaction. The idea here is that when the unit load moves across the beam, its maximum effect on the left-end reaction will be when it is directly lying on the left end support. As the load moves away from the left end support, its influence on the left end reaction will continue to diminish until it gets to the least value of zero, when it is lying directly on the right end support.

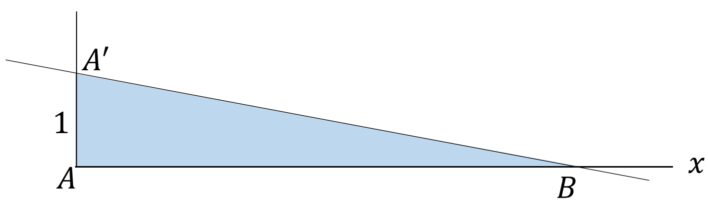


Fig. 9.2. Influence line for R_A.

9.3.1.2 Influence Line for Right End Support Reaction R_B (Fig. 9.3)

(a) At the right end support (point B), plot an ordinate of value +1 (point B').

(b) Draw a line joining point B' and point A.

(c) The triangle $AB'B$ is the influence line for the right end support reaction. The explanation for the influence line for the right end support reaction is similar to that given for the left end support reaction. The maximum effect of the unit load occurs when it is lying directly on the right support. As the load moves away from the right end support, its influence on the support reaction decreases until it is zero, when the load is directly lying on the left support.

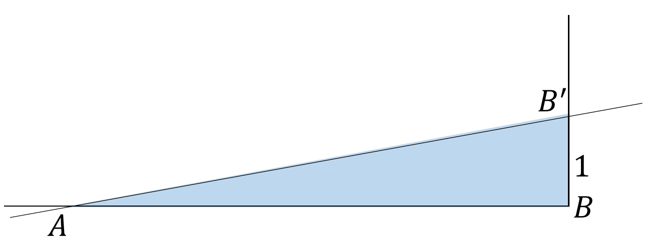


Fig. 9.3. Influence line for R_B.

9.3.1.3 Influence Line for Shearing Force at Section *n*

(a) At the left end support (point A), plot an ordinate equal +1 (point A'), as shown in Figure 9.4b.

(b) Draw a line joining point A' and the zero ordinate at point B.

(c) At the right end support (point B), plot an ordinate equal –1 (point B').

(d) Draw a line joining B' and the zero ordinate at point A.

(e) Drop a vertical line from the section under consideration to cut lines $A'B$ and AB' at points N' and N'', respectively.

(f) The diagram $ABN'N''$ is the influence line of the shear force at the section n.

(g) Use a similar triangle to determine the ordinates n-N' and n-N,'' as follows:

$$\frac{1}{L} = \frac{n-N'}{b} \rightarrow N' = \frac{b}{L}$$

$$\frac{-1}{L} = \frac{n-N''}{a} \rightarrow N'' = -\frac{a}{L}$$

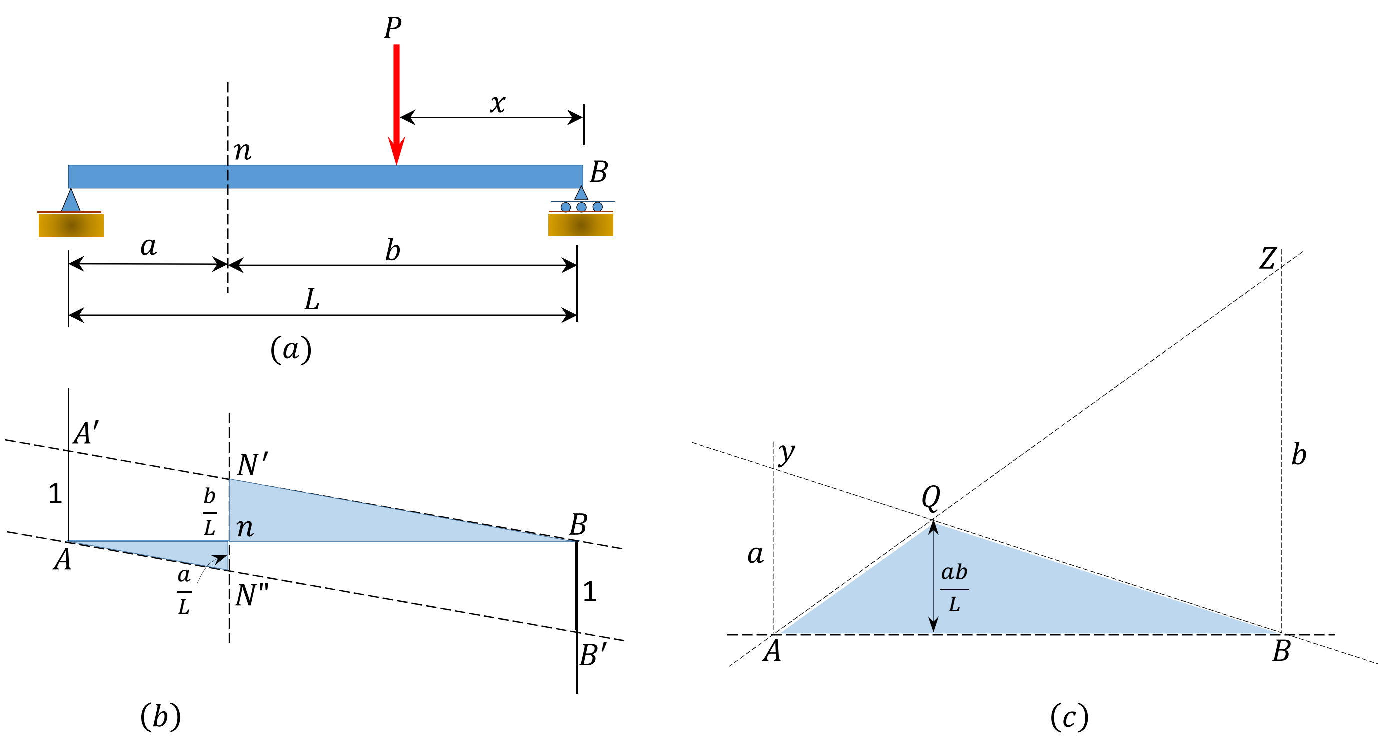


Fig. 9.4. Influence line for shear (b) and moment (c) at secton m.

9.3.1.4 Influence Line for Bending Moment at Section *n*

(a) At the left end support (point *A*), plot an ordinate of a value equal to the distance from the left end support to the section *n*. For example, the distance *a* in Figure 9.4c (denoted as point *Y* in Figure 9.4c).

(b) Draw a line joining point *Y* and the zero ordinate at point *B* at the right end support.

(c) Draw a vertical line passing through section *n* and intersecting the line *AZ* at point *Q*.

(d) Draw a straight line *AQ* connecting *A* and *Q*.

(e) The triangle *AQB* is the influence line for the moment at section *n*. Alternatively, ignore steps (b), and (c) and (d) and go to step (f).

(f) At the right end support (point *B*), plot an ordinate equal +b. For example, the distance from the right end support to the section *n* (denoted as point *Z*).

(g) Draw a line joining *Z* and the zero ordinate at *A* (position of the left end support).

(h) At the left end support (point *A*), plot an ordinate equal +a. For example, the distance from the left end support to the section *n* (denoted point *Y*).

(i) Draw a line joining *Y* and the zero ordinate at *B* (position of the right end support).

(j) Lines *AZ* and *BY* intersect at *Q*.

(k) The triangle *AQB* is the influence line for the moment at section *n*. If accurately drawn, with the right sense of proportionality, the intersection *Q* should lie directly on a vertical line passing through the section *n*.

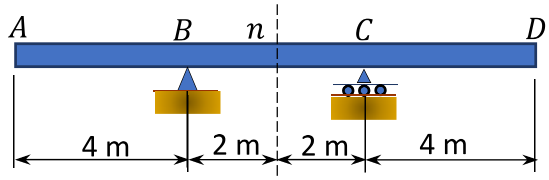
(l) The value of the ordinate *nQ* can be obtained using a similar triangle, as follows:

$$\frac{a}{L} = \frac{nQ}{b} \rightarrow nQ = \frac{ab}{L}$$

or $\frac{b}{L} = \frac{nQ}{a} \rightarrow nQ = \frac{ab}{L}$

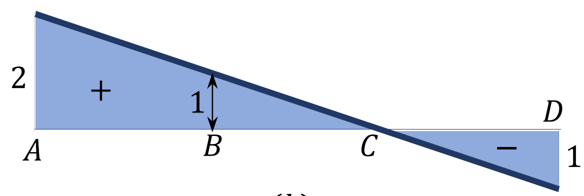
Example 9.1

For the double overhanging beam shown in Figure 9.5a, construct the influence lines for the support reactions at *B* and *C* and the shearing force and the bending moment at section *n*.



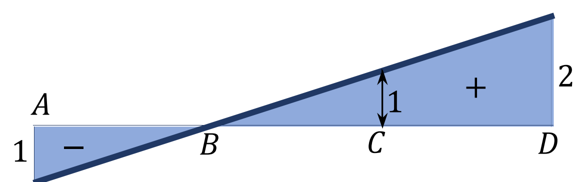
(a)

Fig. 9.5. Double overhanging beam.



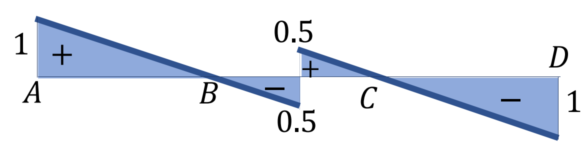
Influence line for reaction B_y

(b)



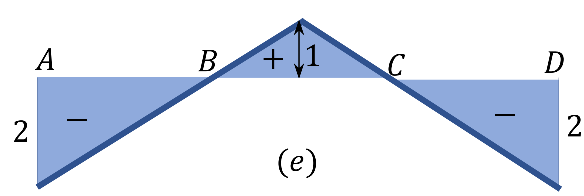
Influence line for reaction C

(c)



Influence line for shear V_n

(d)



Influence line for moment M_n

(e)

Solution

I.L. for B_y.

Step 1. At the position of support *B* (point *B*), plot an ordinate +1.

Step 2. Draw a straight line connecting the plotted point (+1) to the zero ordinate at the position of support *C*.

Step 3. Continue the straight line in step 2 until the end of the overhangs at both ends of the beam. The influence line for B_y is shown in Figure 9.5b.

Step 4. Determine the ordinates of the influence line at the overhanging ends using a similar triangle, as follows:

Ordinate at *A*:

$$\frac{1}{BC} = \frac{x}{AC}; \Rightarrow x = \frac{AC}{BC} = \frac{8}{4} = 2 \text{ m}$$

Ordinate at *D*:

$$\frac{1}{BC} = \frac{x}{CD}; \Rightarrow x = \frac{CD}{BC} = \frac{4}{4} = 1 \text{ m}$$

I.L. for C_y.

Step 1. At the position of support *C* (point *C*), plot an ordinate $+1$.

Step 2. Draw a straight line connecting the plotted point $(+1)$ to the zero ordinate at the position of support *B*.

Step 3. Continue the straight line in step 2 until the end of the overhangs at both ends of the beam. The influence line for B_y is shown in Figure 9.5c.

Step 4. Determine the ordinates of the influence line at the overhanging ends using a similar triangle, as follows:

Ordinate at *D*:

$$\frac{1}{BC} = \frac{x}{BD}; \Rightarrow x = \frac{BD}{BC} = \frac{8}{4} = 2 \text{ m}$$

Ordinate at *A*:

$$\frac{x}{AB} = \frac{1}{BC}; \Rightarrow x = \frac{AB}{BC} = \frac{4}{4} = 1 \text{ m}$$

I.L. for shear Vn.

Step 1. At the position of support *B* (point B), plot an ordinate $+1$.

Step 2. Draw a straight line connecting the plotted point $(+1)$ to the zero ordinate at the position of support *C*. Continue the straight line at *C* until the end of the overhang at end *D*.

Step 3. At the position of support *C* (point *C*), plot an ordinate -1.

Step 4. Draw a straight line connecting the plotted point (-1) to the zero ordinate at the position of support *B*. Continue the straight line at *B* until the end of the overhang at end *A*.

Step 5. Draw a vertical passing through the section whose shear is required to intersect the lines in step 2 and step 3.

Step 6. Connect the intersections to obtain the influence line, as shown in Figure 9.5d.

Step 7. Determine the ordinates of the influence lines at other points by using similar triangles, as previously demonstrated.

I.L. for Moment M_n.

Step 1. At point *B*, plot the ordinate equal +2 m.

Step 2. Draw a straight line connecting the plotted ordinate in step 1 to the zero ordinate in support *C*.

Step 3. At point *C*, plot the ordinate equal +2 m.

Step 4. Draw a straight line connecting the plotted ordinate in step 3 to the zero ordinate at support *B*.

Step 5. Continue the straight lines from the intersection of the lines drawn in steps 2 and 4 through the supports to the overhanging ends, as shown in Figure 9.5e.

Step 6. Determine the values of the influence lines at other points using similar triangles, as previously demonstrated.

Example 9.2

For the beam with one end overhanging support B, as shown in Figure 9.6, construct the influence lines for the bending moment at support *B*, the shear force at support *B*, the support reactions at *B* and *C*, and the shearing force and the bending moment at a section "*k*."

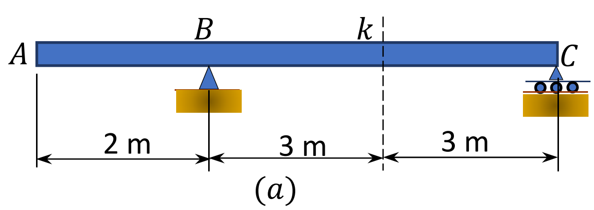
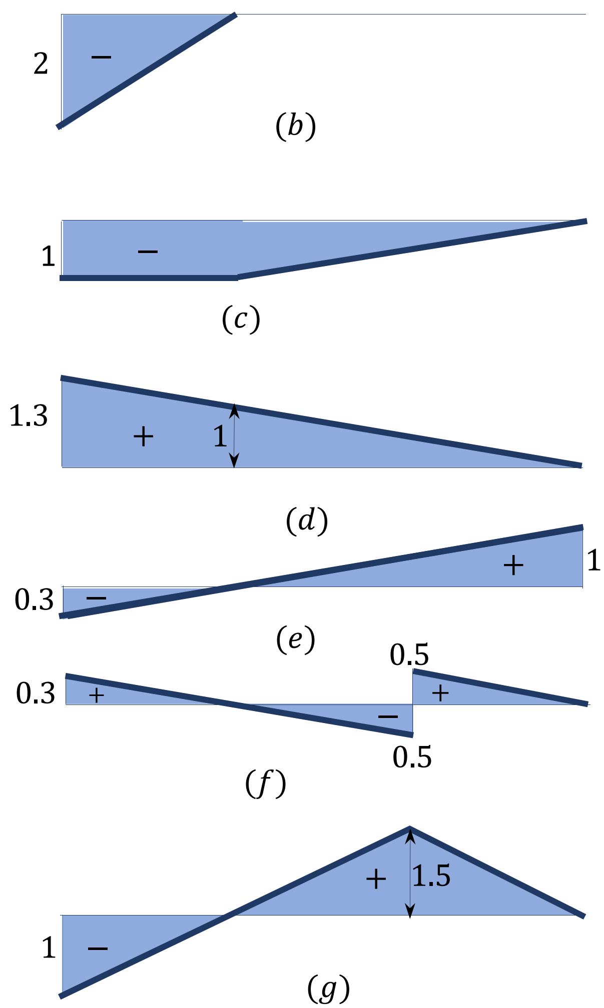


Fig. 9.6. Beam with one overhanging support.



Influence line for moment M_B

Influence line for shear V_B

Influence line for reaction B_y

Influence line for reaction C

Influence line for shear V_k

Influence line for shear M_k

Solution

The influence lines in example 9.2 for the desired functions were constructed based on the procedure described in the previous section and example.

9.3.2 Compound Beams

To correctly draw the influence line for any function in a compound beam, a good understanding of the interaction of the members of the beam is necessary, as was discussed in chapter 3, section 3.3. The student should recall from the previous section that a compound beam is made up of the primary structure and the complimentary structure. The two facts stated below must always be remembered, since the extent of the spread of the influence line of compound beams depends on them. Remembering these facts will also serve as a temporary check to ascertain the correctness of the drawn influence line.

The moving unit load will have an effect on the functions of the primary structure when it is located at any point, not only on the primary structure but also on the complimentary structure, since the latter constitutes a loading on the former.

The moving unit load will have effect only on the functions of the complimentary structure when it is located within the complimentary structure; it will not have an effect on any function of the complimentary structure when it is at any point on the primary structure.

The afore-stated facts will be demonstrated in the following examples.

Example 9.3

For the compound beam shown in Figure 9.7, construct the influence lines and indicate the critical ordinates for the support reactions at *A*, *B*, and *D*, the bending moment at *B*, and the shear at hinge *C*.

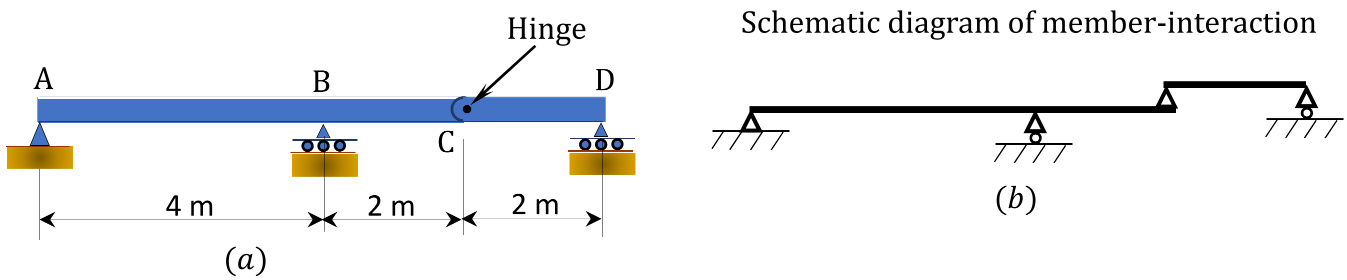
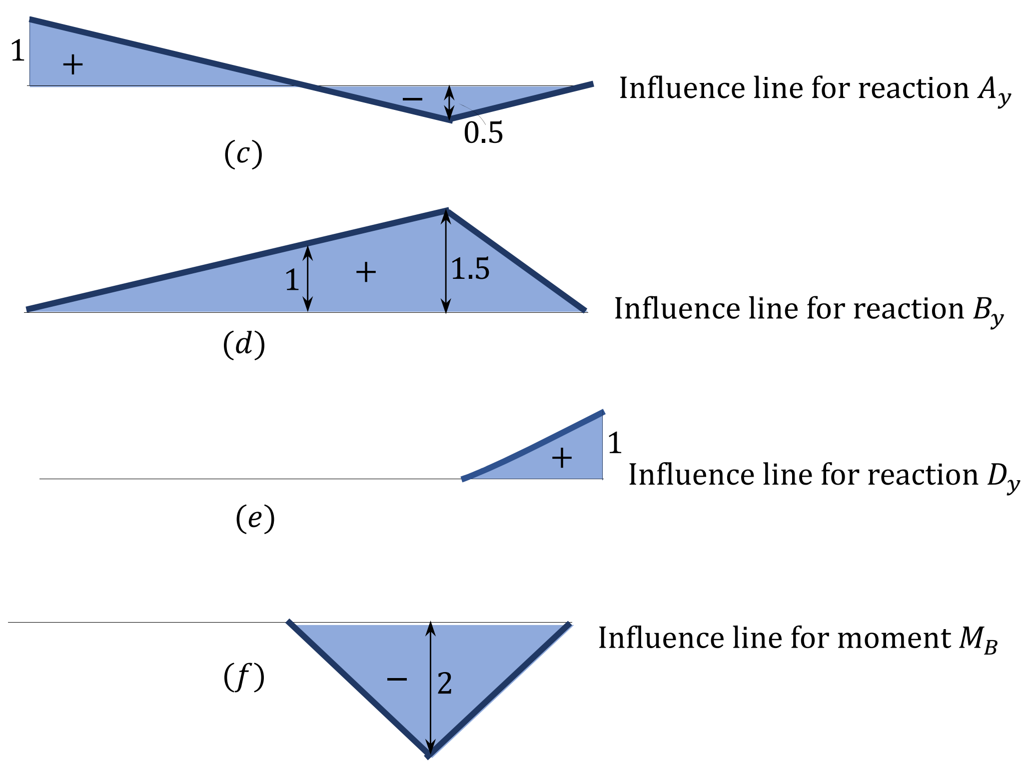


Fig. 9.7. Compound beam.



Influence line for shear V_c at hinge C

(g)

Solution

Prior to the construction of the influence lines for desired functions, it is necessary to first observe the extent of the influence lines through the schematic diagram of member-interaction, as shown in Figure 9.7b.

I.L. for A_y. The reaction A_y is a function in the primary structure, so the unit load will have influence on this function when it is located at any point on the beam, as was previously stated in section 9.3.2. With this understanding, construct the influence line of A_y, as follows:

Step 1. At point *A*, plot an ordinate $+1$.

Step 2. Draw a straight line connecting the plotted ordinate in step 1 to the zero ordinate in support *B* and continue this line until the end of the overhanging end of the primary structure, as shown in the interaction diagram.

Step 3. Draw a straight line connecting the ordinate at the end of the overhang to the zero ordinate at support *D*. The influence line is as shown in Figure 9.7c.

Step 4. Use a similar triangle to compute the ordinates of the influence line

I.L. for B_y. The influence line for this reaction will cover the entire length of the beam because it is a support reaction in the primary structure. With this knowledge, construct the influence line for B_y, as follows:

Step 1: At point *B*, plot an ordinate $+1$.

Step 2. Draw a straight line connecting the plotted ordinate in step 1 to the zero ordinate in support *A*. Continue the line in support *B* until the end of the overhanging end of the primary structure, as shown in the interaction diagram.

Step 3. Draw a straight line connecting the ordinate at the overhanging end to the zero ordinate at support *D*. The influence line for B_y is shown in Figure 9.7d.

Step 4. Use a similar triangle to determine the values of the ordinate of the influence line.

I.L. for D_y. The reaction D_y is a function in the complimentary structure and will be influenced when the unit load lies at any point along the complimentary structure. It will not be influenced when the unit load transverses the primary structure, as was stated in section 9.3.2. Thus, the extent of the influence line will be the length of the complimentary structure. Knowing this, draw the influence line for D_y.

Step 1. At point *D*, plot the ordinate +1.

Step 2. Draw a straight line connecting the plotted ordinate in step 1 to the zero ordinate at hinge *C*. The influence line for D_y is as shown in Figure 9.7e.

The influence lines for the moment at *B* and the shear *C* are shown in Figure 9.7f and Figure 9.7g, respectively.

Example 9.4

For the compound beam shown in Figure 9.8a, construct the influence lines and indicate the critical ordinates for the support reactions at *F* and *G*, the shear force and bending moment at *D*, and the moment at *F*.

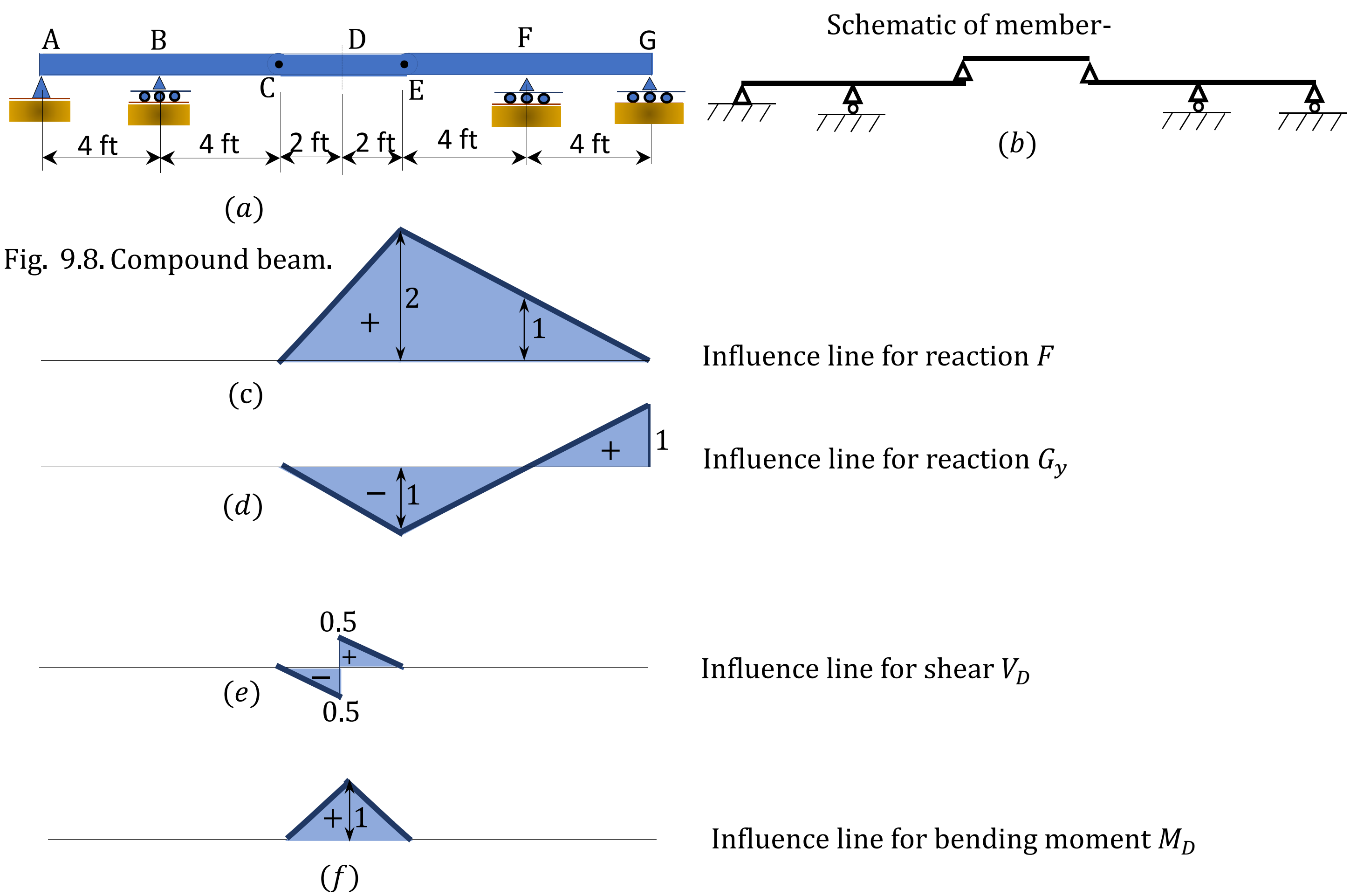
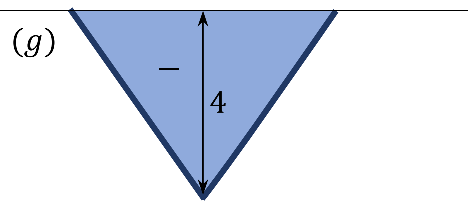


Fig. 9.8. Compound beam.



Influence line for bending moment M_F

Solution

Shown in Figure 9.8c through Figure 9.8g are the influence lines for the desired functions. The schematic diagram of the member interaction shown in Figure 9.8b immeasurably aids the initial perception of the range of the influence line of each function. Construction of the influence lines follows the description outlined in the previous sections.

9.3.3 Influence Lines for Girders Supporting Floor Systems

Thus far, the examples and text have only considered cases where the moving unit load is applied directly to the structure. But, in practice, this may not always be the case. For instance, sometimes loads from building floors or bridge decks are transmitted through secondary beams, such as stringers and cross beams to girders supporting the building or bridge floor system, as shown in Figure 9.9. Columns, piers, or abutments in turn support the girders.

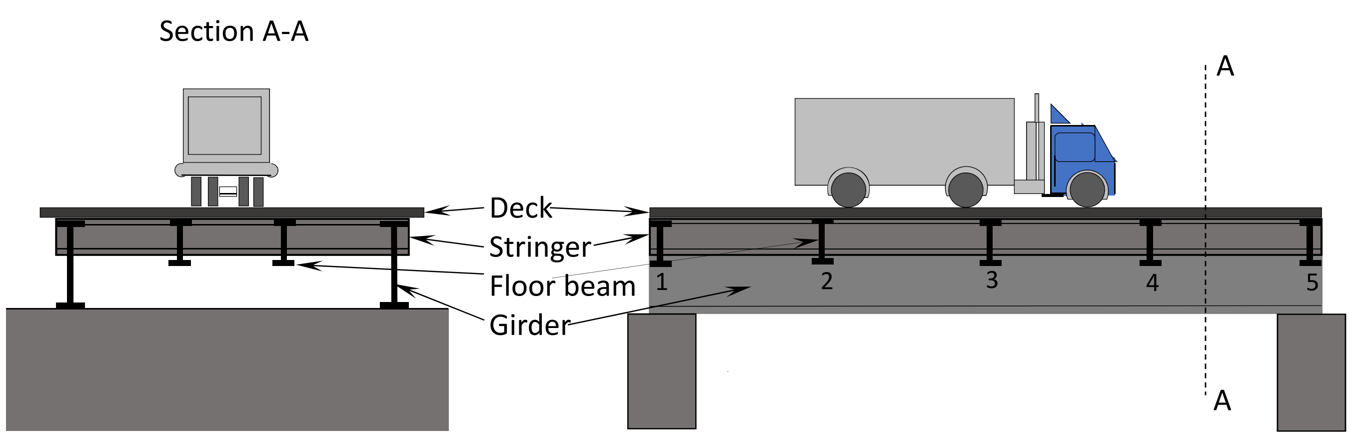
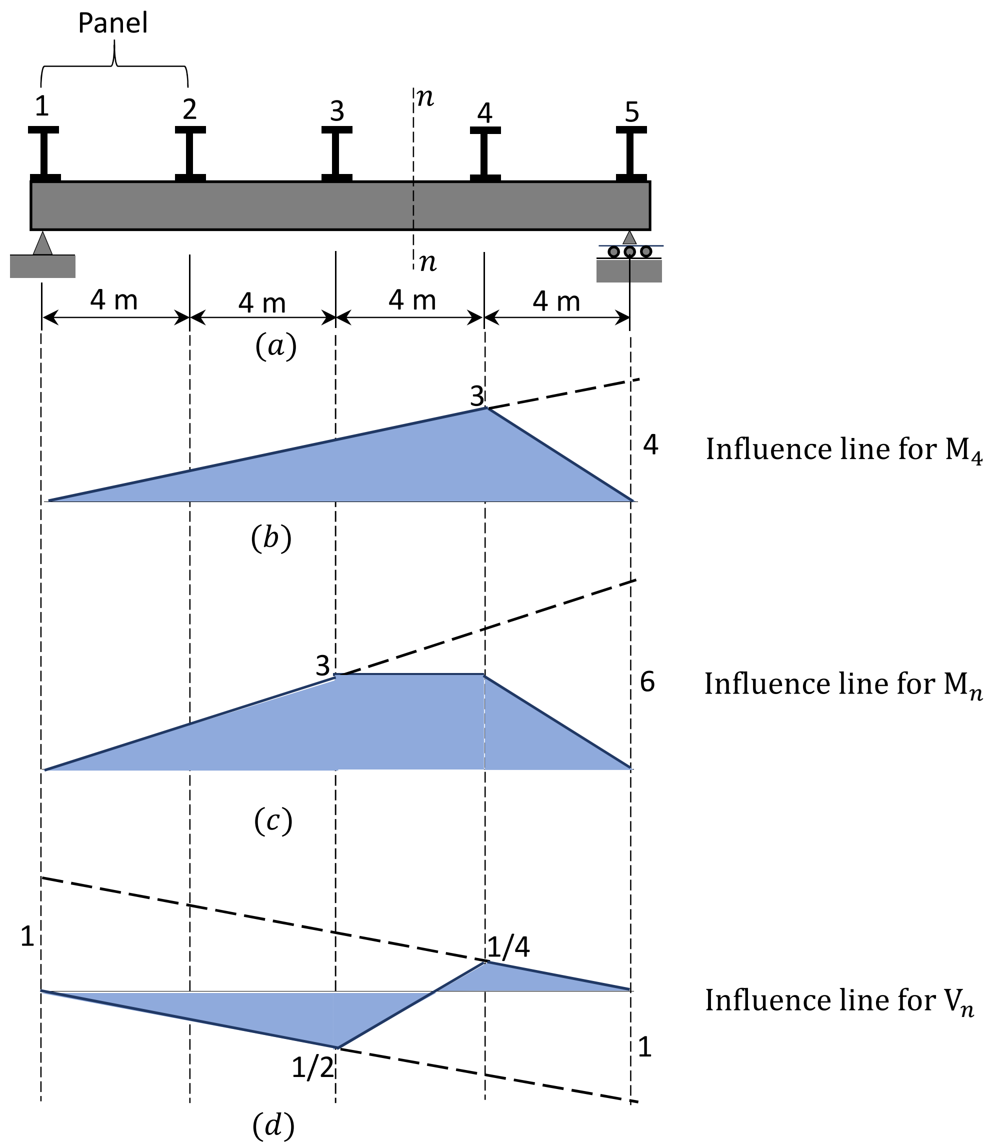


Fig. 9.9. Transfer of load to girder by system of stringers and floor beams.

As shown in Figure 9.9, the vehicular load from the bridge deck is transferred to the girder at points 1, 2, 3, 4, and 5, referred to as panel points, where the floor beams are in contact with the girder. The segment between two successive contact points is known as a panel. For an illustration of the construction of influence lines in a case of indirect application of loads, the floor beams and the girder of Figure 9.9 are separated from the entire system, as shown in Figure 9.10. Assume the length of each panel equals 4m. Construct the influence lines for the moment at point 4 and for the moment and shear at a section *n* at the midpoint of 3 and 4 (a point lying within panel 3-4). The influence line for the moment at point 4 is shown in Figure 9.10b; notice that the construction of the influence line for moment at this point is exactly like the cases considered in previous sections, where the moving load is applied directly to the beam. When the unit load moves to the right of 4 and to the left of 3, the influence line for the moment for any section within panel 3-4 will be constant, as shown in Figure 9.10c. The construction of the influence line for the shear of any section within the panel 3-4 is obtained in the same manner as when the unit load is directly applied to the girder, with the exception that a diagonal line is drawn to connect the points where a vertical line drawn from the points intersect with the construction line.

Fig. 9.10. Influence lines in a case of indirect application of loads.



Example 9.5

Draw the influence lines for the moment at C and the shear in panel BC of the floor girder shown in Figure 9.11.

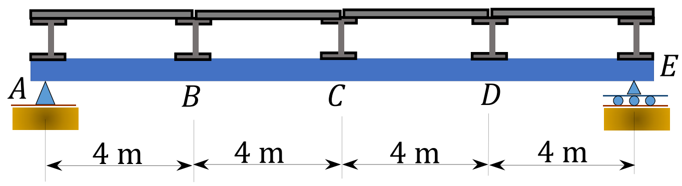


Fig. 9.11a. Floor girder.

Solution

Influence line for M_C. To obtain the values of the influence line of M_C, successively locate a load of 1 kN at panel points A, B, C, D, and E. To determine the moment, use the equation of statics. The values of M_C at the respective panel points are presented in Table 9.1. When the unit load is located at B, as shown in Figure 9.11b, the value of M_C is determined as follows:

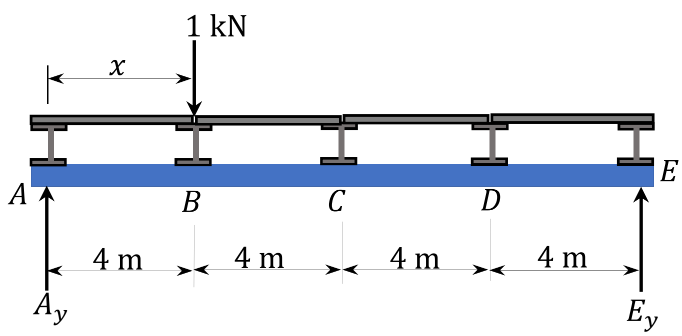


Fig. 9.11b. Unit load at B.

First, determine the support reactions in the beam using the equation of static equilibrium.

$$+\circlearrowleft \sum M_E = 0 \qquad -A_y(16) + 1(12) = 0 \qquad A_y = 0.75 \text{ kN}$$

$$+\uparrow \sum F_y = 0 \qquad 0.75 + E_y - 1 = 0 \qquad E_y = 0.25 \text{ kN}$$

Then, using the computed reaction, determine M_C, as follows:

$$M_c = 0.25(12) = 3 \text{ kN} - \text{m}$$

Table 9.1. The values of M_C at the respective panel points.

x(m)	Reactions(kN) A_y	E_y	M_C(kN. m)
0	1	0	0
4	0.75	0.25	$0.25(8) = 2$
8	0.5	0.5	$0.5(8) = 4$
12	0.25	0.75	$0.25(8) = 2$
16	0	1	0

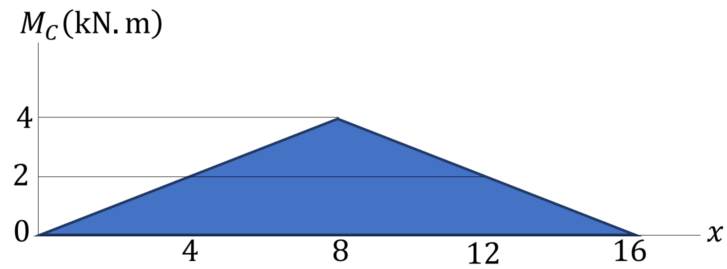


Fig. 9.11c. Influence line for M_C.

Influence line for V_{BC}. To obtain the values of the influence line of MV_{BC}, a load of 1 kN is successively located at panel points A, B, C, D, and E. To determine the shear force, use the equation of statics. The values of V_C at the respective panel points are presented in Table 9.2.

Table 9.2. The values of V_C at the respective panel points.

x(m)	Reactions(kN) A_y	E_y	V_{BC}(kN)
0	1	0	0
4	0.75	0.25	−0.25
8	0.5	0.5	0.5
12	0.25	0.75	0.25
16	0	1	0

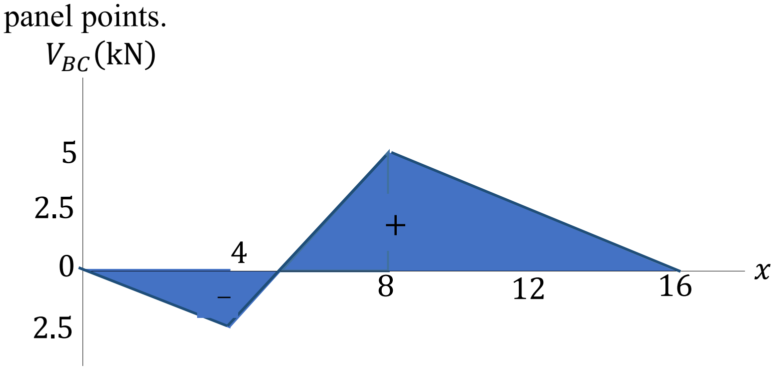


Fig. 9.11d. Influence line for V_{BC}.

9.3.4 Influence Lines for Trusses

The procedure for the construction of influence lines for truss members is similar to that of a girder supporting a floor system considered in section 9.3.3. Loads can be transmitted to truss members via the top or bottom panel nodes. In Figure 9.12 the load is transmitted to members through the top panel nodes. As the live loads move across the truss, they are transferred to the top panel nodes by cross beams and stringers. The influence lines for axial forces in truss members can be constructed by connecting the influence line ordinates at the panel nodes with straight lines.

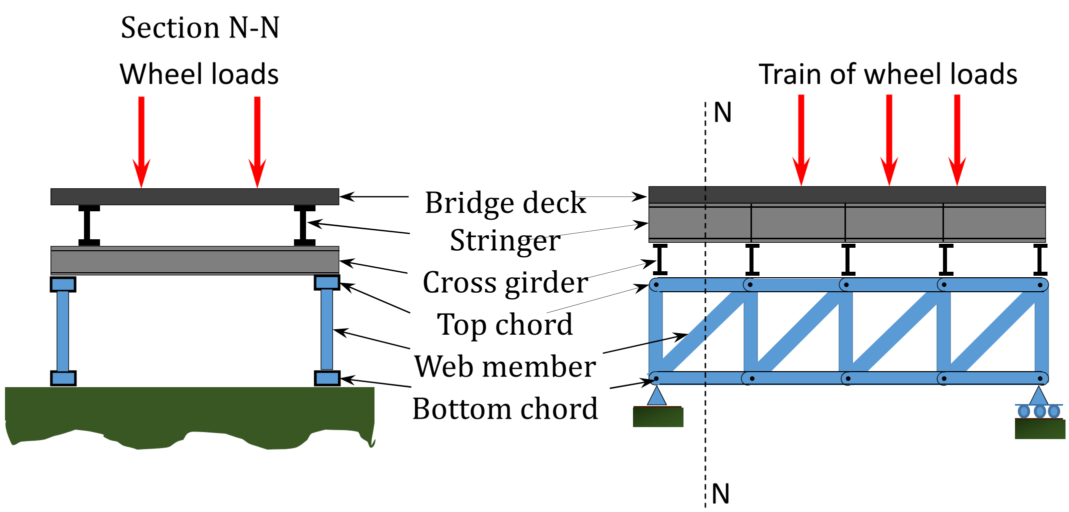


Fig. 9.12. Load transfered by system of stringers and cross beams.

To illustrate the procedure for the construction of influence lines for trusses, consider the following examples.

Example 9.6

Draw the influence lines for the reactions A_y, F_y, and for axial forces in members *CD*, *HG*, and *CG* as a unit load moves across the top of the truss, as shown in Figure 9.13.

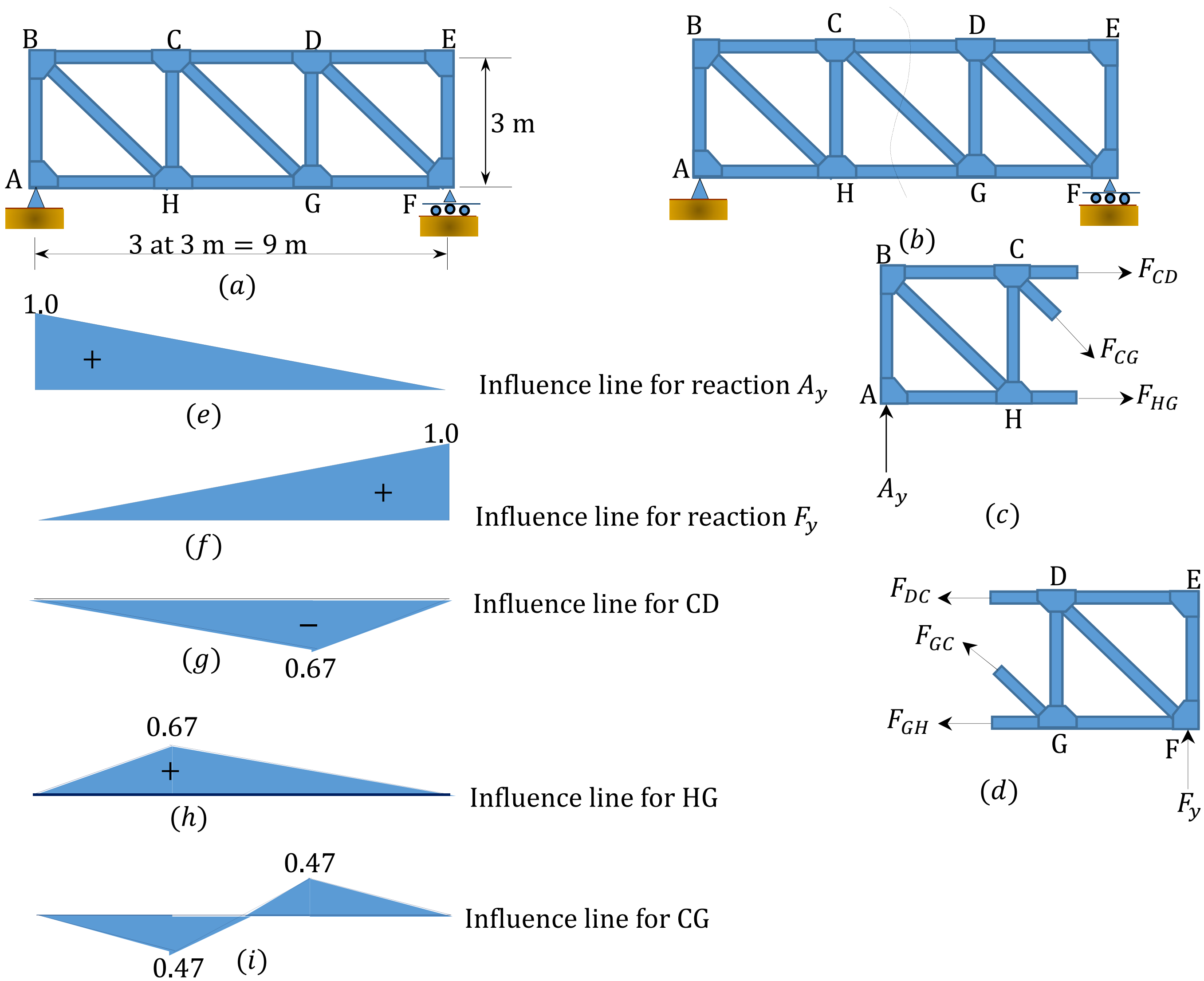


Fig. 9.13. Truss.

Solution

The drawing of the influence lines for trusses is similar to that of a beam. The first step towards drawing the influence lines for the axial forces in the stated members is to pass an imaginary section through the members, as shown in Figure 9.13b, and to apply equilibrium to the part on either side of the section. The step-by-step procedure for drawing the influence line for each of the members is stated below.

Influence line for the axial force in member *CD*. When the unit load is situated at any point to the right of *D*, considering the equilibrium of the left segment *AH* (Fig. 9.13c), it suggests the following:

$$+\circlearrowleft \sum M_G = 0 \qquad -A_y(6) - F_{CD}(3) = 0 \qquad F_{CD} = -2A_y$$

The obtained expression of F_{CD} in terms of A_y is indicative of the fact that the influence line for F_{CD} in the portion *DE* can be determined by multiplying the corresponding portion of the influence line for the reaction A_y by -2. The influence line for A_y is shown in Figure 9.13e.

When the unit load is situated at any point to the left of *C*, considering the equilibrium of the right segment *GF* (Fig. 9.13d), it suggests the following:

$$+\circlearrowleft \sum M_G = 0 \qquad F_y(3) + F_{CD}(3) = 0 \qquad F_{CD} = -F_y$$

The obtained expression of F_{CD} in terms of F_y is indicative of the fact that the influence line for F_{CD} in the portion *AH* can be determined by multiplying the corresponding portion of the influence line for the reaction F_y by -1. The influence line for F_y is shown in Figure 9.13f.

The influence line of the axial force in member *CD* constructed from the influence lines for the reactions A_y and F_y is shown in Figure 9.13g.

Influence line for member *HG*. When the unit load is situated at any point to the right of *D*, considering the equilibrium of the left segment *AH* (Fig. 9.13c), it suggests the following:

$$+\circlearrowleft \sum M_c = 0 \qquad -A_y(3) + F_{HG}(3) = 0 \qquad F_{HG} = A_y$$

The obtained expression of F_{HG} in terms of A_y implies that the influence line for F_{HG} in the portion *DE* is identical to that of A_y within the corresponding segment.

When the unit load is situated at any point to the left of *C*, considering the equilibrium of the right segment *GF* (Fig. 9.13d), it suggests the following:

$$+\circlearrowleft \sum M_c = 0 \qquad F_y(6) - F_{HG}(3) = 0 \qquad F_{HG} = 2F_y$$

The obtained expression of F_{HG} in terms of F_y is indicative of the fact that the influence line for F_{HG} in the portion *AH* can be determined by multiplying the corresponding portion of the influence line for the reaction F_y by 2.

The influence line of the axial force in member *HG* constructed from the influence line for the reactions A_y and F_y is also shown in Figure 9.13h.

Influence line for the axial force in member *CG*. When the unit load is situated at any point to the right of *D*, considering the equilibrium of the left segment *AH* (Fig. 9.13C), it suggests the following:

$$+\uparrow \sum F_y = 0 \qquad A_y - F_{CG} \cos 45^\circ = 0$$

$$F_{CG} = \frac{A_y}{\cos 45^\circ} = 1.41 A_y$$

The obtained expression of F_{CG}, with reference to A_y, implies that the influence line for F_{CG} in the portion *DE* can be determined by multiplying the corresponding portion of the influence line for the reaction A_y by 1.41.

When the unit load is situated at any point to the left of *C*, considering the equilibrium of the right segment *GF* (Fig. 9.13d), it suggests the following:

$$F_y + F_{CG} \cos 45^\circ = 0$$

$$F_{CG} = -\frac{F_y}{\cos 45^\circ} = -1.41 F_y$$

The obtained expression of F_{CG} in terms of F_y is indicative of the fact that the influence line for F_{CG} in the portion *AH* can be determined by multiplying the corresponding portion of the influence line for the reaction F_y by -1.41.

The influence line of the axial force in member *CG* constructed from the influence line for the reactions A_y and F_y is shown in Figure 9.13i.

Example 9.7

Draw the influence lines for the force in member *CH* as a unit load moves across the top of the truss, as shown in Figure 9.14a.

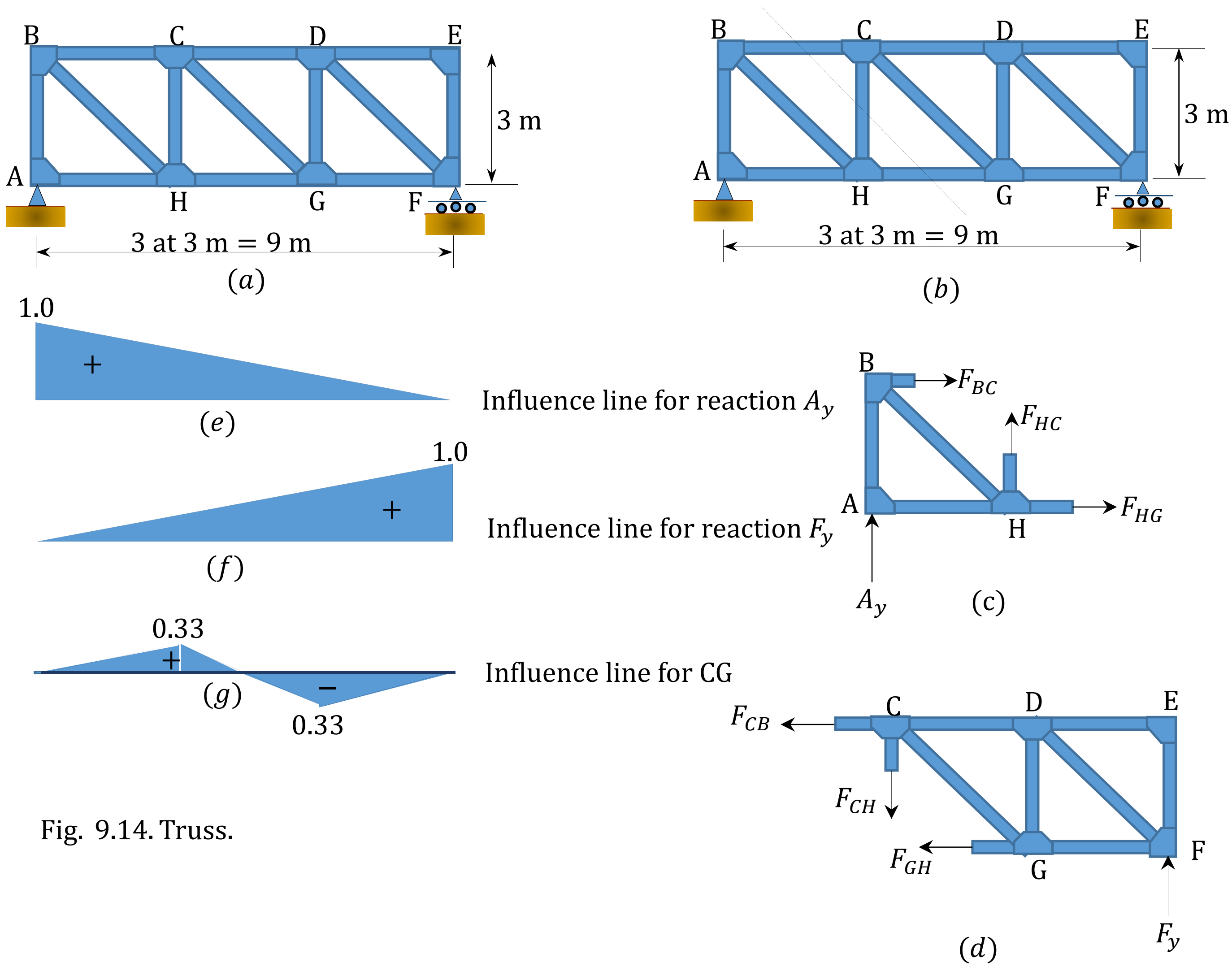


Fig. 9.14. Truss.

Solution

To obtain the expression for the influence line for the axial force in member *CH*, first pass an imaginary section that cuts through this member, as shown in Figure 9.14a.

When the unit load is situated at any point to the right of *G*, considering the equilibrium of the left segment *AH* (Fig. 9.14 C), it suggests the following:

$$+\uparrow \sum F_y = 0 \qquad A_y + F_{CH} = 0$$

$$F_{CH} = -A_y$$

The obtained expression of F_{CH} in terms of A_y indicates that the influence line for F_{CH} in the portion *AH* can be determined by multiplying the corresponding portion of the influence line for the reaction A_y by -1.

When the unit load is located at any point to the left of *H*, considering the equilibrium of the right segment *GF* (Fig. 9.14d), it suggests the following:

$$+\uparrow \sum F_y = 0 \quad F_y - F_{CH} = 0$$
$$F_{CH} = F_y$$

The obtained expression of F_{CH} in terms of F_y implies that the influence line for F_{CH} in the portion *GF* is identical to that of F_y within the corresponding segment.

The influence line of *CG* is shown in Figure 9.14g.

9.4 Uses of Influence Lines

9.4.1 Uses of Influence Lines to Determine Response Functions of Structures Subjected to Concentrated Loads

The magnitude of a response function of a structure due to concentrated loads can be determined as the summation of the product of the respective loads and the corresponding ordinates of the influence line for that response function. Example 9.5 and example 9.6 illustrate such cases.

Example 9.8

A simple beam is subjected to three concentrated loads, as shown in Figure 9.15a. Determine the magnitudes of the reactions and the shear force and bending moment at the midpoint of the beam using influence lines.

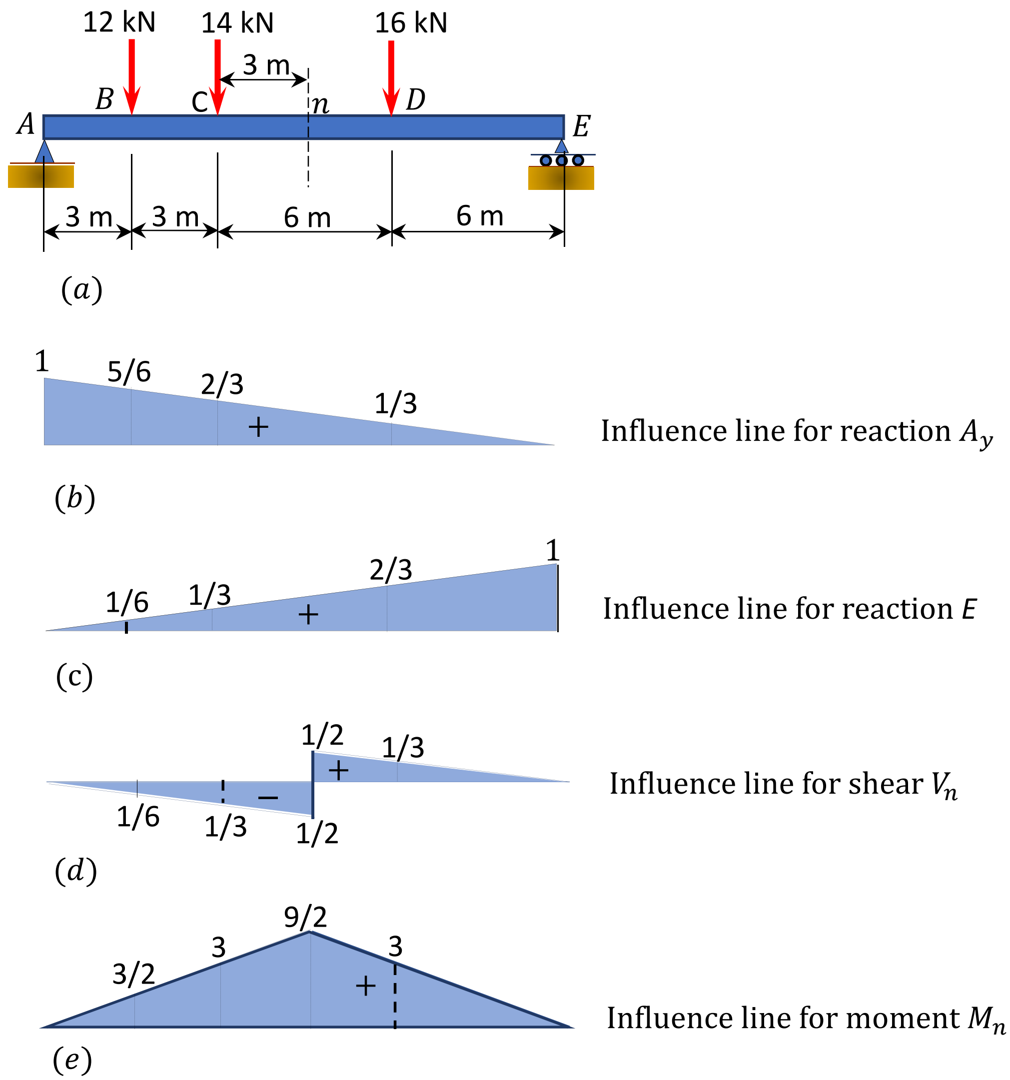


Fig. 9.15. Simple beam.

Solution

First, draw the influence line for the support reactions and for the shearing force and the bending moment at the midpoint of the beam (see Fig. 9.15b, Fig. 9.15c, Fig. 9.15d, and Fig. 9.15e). Once the influence lines for the functions are drawn, compute the magnitude of the response functions, as follows:

Magnitude of the support reactions using the influence line diagrams in Figure 9.15b and Figure 9.15c.

$$A_y = (12)\left(\tfrac{5}{6}\right) + (14)\left(\tfrac{2}{3}\right) + (16)\left(\tfrac{1}{3}\right) = 24.67 \text{ kN}$$

$$E_y = (12)\left(\tfrac{1}{6}\right) + (14)\left(\tfrac{1}{3}\right) + (16)\left(\tfrac{2}{3}\right) = 17.33\ \text{kN}$$

Magnitude of the shear force at section *n* using the influence line diagram in Figure 9.15d.

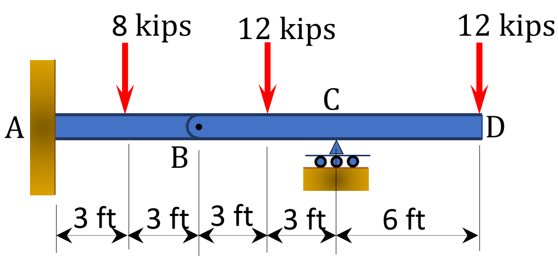
$$V_n = (12)\left(-\tfrac{1}{6}\right) + (14)\left(-\tfrac{1}{3}\right) + (16)\left(\tfrac{1}{3}\right) = -1.33\ \text{kN}$$

Magnitude of the bending moment at section *n* using the influence line diagram of Figure 9.15e.

$$M_n = (12)\left(\tfrac{3}{2}\right) + (14)(3) + (16)(3) = 108\ \text{kN.m}$$

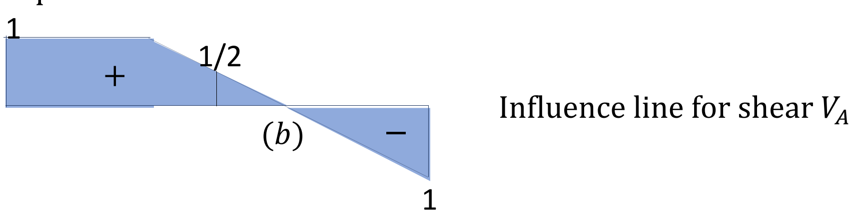
Example 9.9

A compound beam is subjected to three concentrated loads, as shown in Figure 9.16a. Using influence lines, determine the magnitudes of the shear and the moment at *A* and the support reaction at *D*.

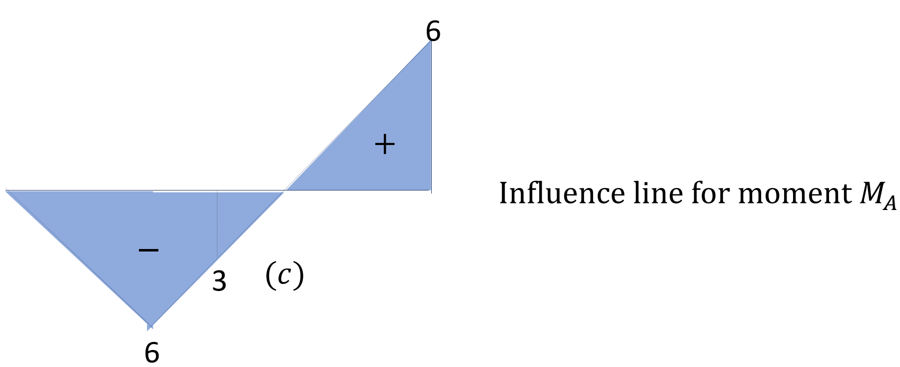


(a)

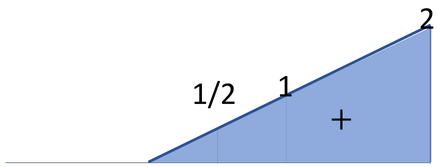
Fig. 9.16. Compound beam.



(b)



(c)



(d)

Influence line for reaction C

Solution

First, draw the influence line for the shear force V_A, bending moment M_A, and reaction C_y. The influence lines for these functions are shown in Figure 9.16b, Figure 9.16c, and Figure 9.16d. Then, compute the magnitude of these response functions, as follows:

The magnitude of the shear at section *n* using the influence line diagram in Figure 9.16b.

$$V_A = (8)(1) + (12)\left(\tfrac{1}{2}\right) + -(12)(1) = 26 \text{ kips}$$

The magnitude of the bending moment at section *n* using the influence line diagram in Figure 9.16c.

$$M_A = (8)(-6) + (12)(-3) + (12)(6) = -12 \text{ kip.ft}$$

Magnitude of the support reaction C_y using the influence line diagram in Figure 9.16d.

$$C_y = (12)\left(\tfrac{1}{2}\right) + (12)(2) = 30 \text{ kips}$$

9.4.2 Uses of Influence Lines to Determine Response Functions of Structures Subjected to Distributed Loads

The magnitude of a response function of a structure subjected to distributed loads can be determined as the product of the intensity of the distributed load and the area of the influence line. Consider a beam subjected to a uniform load ω_x, as shown in Figure 9.17a. First, convert the uniform load to an equivalent concentrated load. The equivalent elementary concentrated load for a distributed load acting on a differential length dx is as follows:

$$dP = \omega_x dx \qquad (9.5)$$

The magnitude of the response function (rf) due to the elementary concentrated load acting on the structure can be expressed as follows:

$$rf = (\omega_x dx)(y) \quad (9.6)$$

where

y = the ordinate of the influence line at the point of application of the load dP.

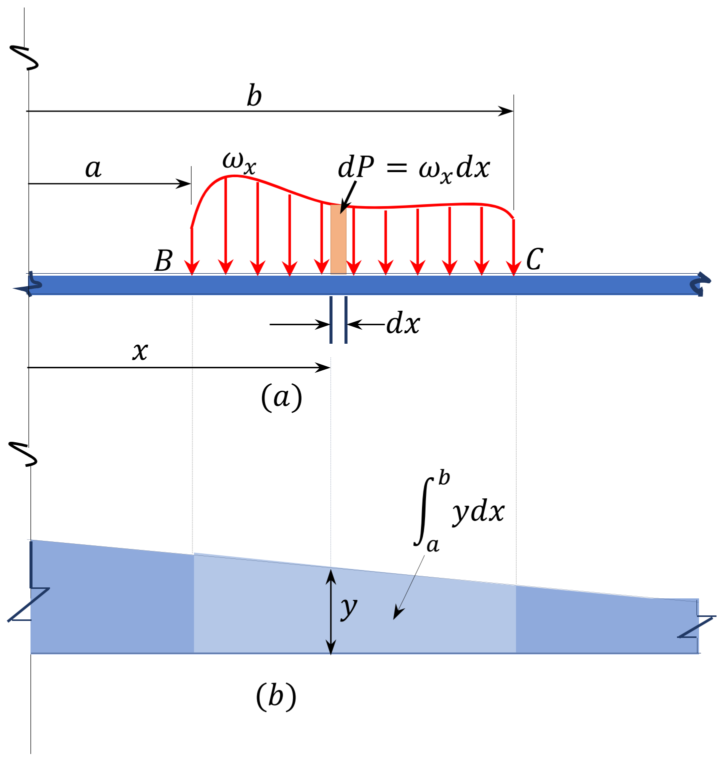


Fig. 9.17. Beam subjected to uniform load.

The total response function (RF) due to the distributed load acting at the segment BC of the beam is obtained by integration, as follows:

$$RF = \int_B^C (\omega_x)(y)dx = \omega_x \int_B^C (y)dx \quad (9.7)$$

The integral $\int_B^C (y)dx$ is the area under the portion of the influence line corresponding to the loaded segment of the beam (see the shaded area in Fig. 9.17b).

Example 9.10

Using influence lines, determine the shear force and the bending moment at the midpoint of the loaded simple beam, as shown in Figure 9.18a.

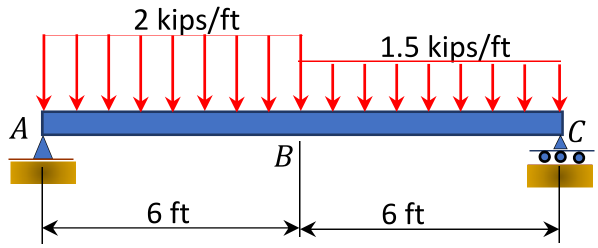
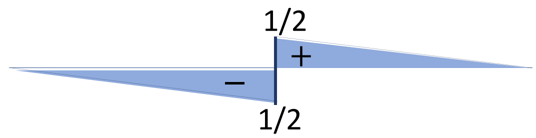
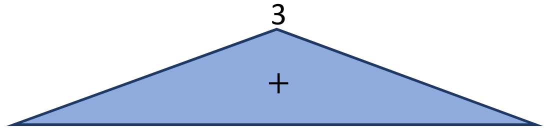


Fig. 9.18. Loaded simple beam. (a)



Influence line for V_B

(b)



Influence line for M_B

(c)

Solution

First, draw the influence line for the shear force and bending moment at the midspan of the beam. The influence lines for these functions are shown in Figure 9.18b and Figure 9.18c. Then, compute the magnitude of these response functions, as follows:

From the influence line diagram, as shown in Figure 9.18b, the magnitude of the shear at B is as follows:

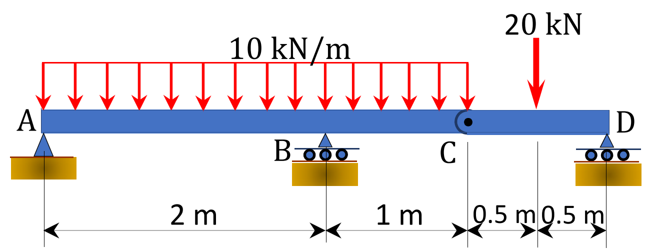
$$V_B = (2)\left(-\tfrac{1}{2} \times 6 \times \tfrac{1}{2}\right) + (1.5)\left(\tfrac{1}{2} \times 6 \times \tfrac{1}{2}\right) = -0.75 \text{ kip}$$

The magnitude of the bending moment at point B, using influence line diagram in Figure 9.18c, is as follows:

$$M_B = (2)\left(\tfrac{1}{2} \times 6 \times 3\right) + (1.5)\left(\tfrac{1}{2} \times 6 \times 3\right) = 31.5 \text{ kip.ft}$$

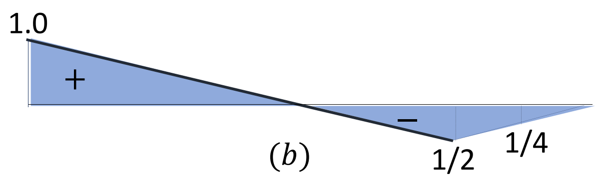
Example 9.11

A compound beam is subjected to a combined loading, as shown in Figure 9.19a. Using influence lines, determine the magnitudes of the reactions at supports A, B, and C.



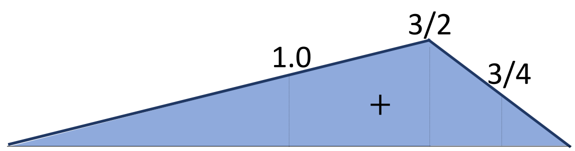
(a)

Fig. 9.19. Compound beam subjected to combined loading.



(b)

Influence line for reaction A_y



(c)

Influence line for reaction B

(d)

Influence line for reaction D

Solution

The magnitude of the support reaction A_y, using the influence line diagram in Figure 9.19b.

$$A_y = (10)\left(\tfrac{1}{2}\right)(2)(1) + (10)\left(\tfrac{1}{2}\right)(1)\left(-\tfrac{1}{2}\right) + (20)\left(-\tfrac{1}{4}\right) = 2.5 \text{ kN}$$

The magnitude of the support reaction B_y, using the influence line diagram in Figure 9.19c.

$$B_y = (10)\left(\tfrac{1}{2}\right)(3)\left(\tfrac{3}{2}\right) + (20)\left(\tfrac{3}{4}\right) = 37.5 \text{ kN}$$

The magnitude of the support reaction D_y, using the influence line diagram of Figure 9.19d.

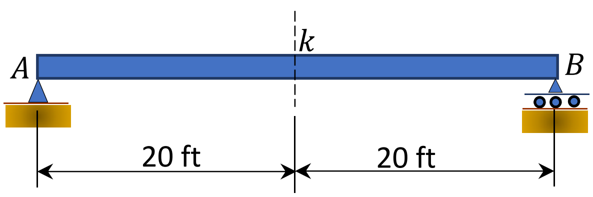
$$D_y = (20)\left(\tfrac{1}{2}\right) = 10 \text{ kN}$$

9.4.3 Use of Influence Lines to Determine the Maximum Effect at a Point Due to Moving Concentrated Loads

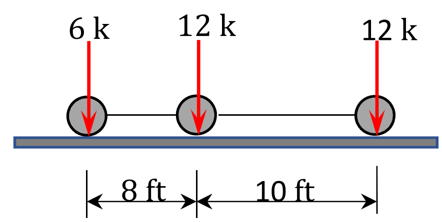
In the analysis and design of structures, such as bridges and cranes subjected to moving loads, it is often desirable to find the position of the moving load(s) that will produce a maximum influence at a point. For some structures, this can be determined by mere inspection, while for most others it may require a trial-and-error process using influence lines. Examples 9.12 and 9.13 illustrate the trial-and-error process involved when using influence lines to compute the magnitude of certain functions of a beam subjected to a series of concentrated moving loads.

Example 9.12

Using influence lines, determine the shear force and bending moment at the midpoint k of a beam shown in Figure 9.20a. The beam is subjected to a series of moving concentrated loads, which are shown in Figure 9.20b.

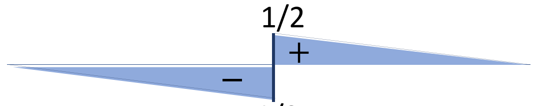


(a)



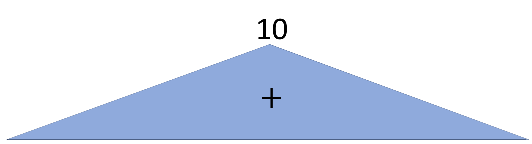
(b)

Fig. 9.20. Beam.



Influence line for shear V_k

(c)



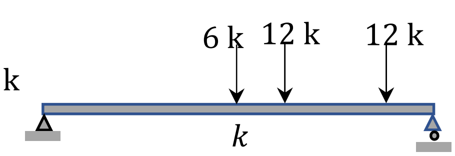
Influence line for moment M_k

(d)

Solution

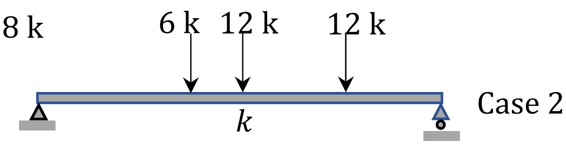
Maximum shear V_k from Figure 9.20c.

Case 1: $V_k = (6)\left(\frac{1}{2}\right) + (12)\left(\frac{3}{10}\right) + (12)\left(\frac{1}{20}\right) = 7.2\text{ k}$



Case 1

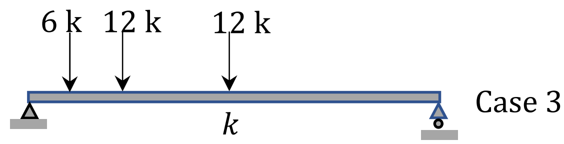
Case 2: $V_k = (6)\left(-\frac{3}{10}\right) + (12)\left(\frac{1}{2}\right) + (12)\left(\frac{3}{10}\right) = 7.8 \text{ k}$



Case 3: $V_k = (6)\left(-\frac{1}{20}\right) + (12)\left(-\frac{1}{4}\right) + (12)\left(-\frac{1}{2}\right) = -9.3 \text{ k}$

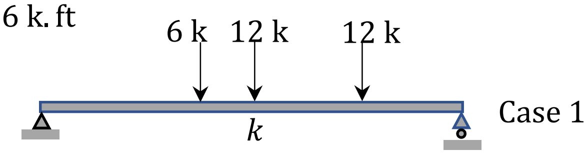
Maximum positive shear = 7.8 k

Maximum negative shear = 9.3 k



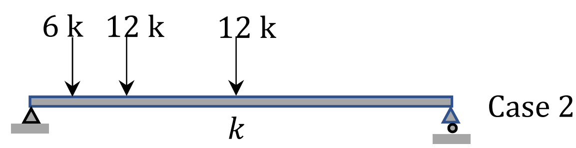
Maximum moment M_k from Figure 9.20d.

Case 1: $M_k = (6)(6) + (12)(10) + (12)(5) = 216 \text{ k.ft}$



Case 2: $M_k = (6)(1) + (12)(5) + (12)(10) = 186 \text{ k.ft}$

Maximum moment $M_k = 216 \text{ k.ft}$



Example 9.13

A compound beam shown in Figure 9.21a is subjected to a series of moving concentrated loads, which are shown in Figure 9.21b. Using influence lines, determine the magnitudes of the reactions at supports A, B, and C and the bending moment at section n.

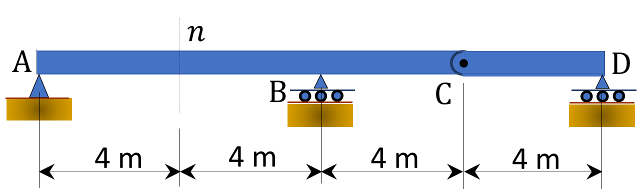
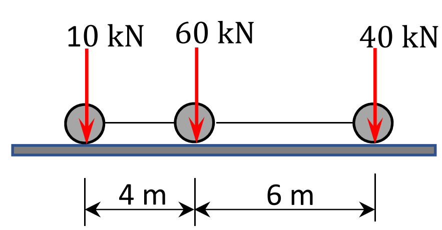
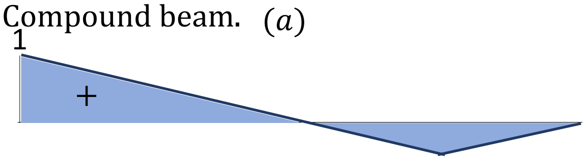


Fig. 9.21. Compound beam. (a)

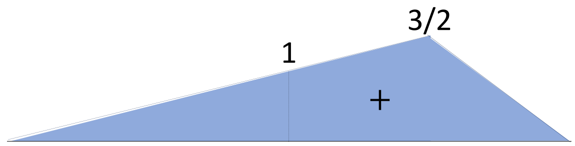


(b)



(c)

Influence line for reaction A_y

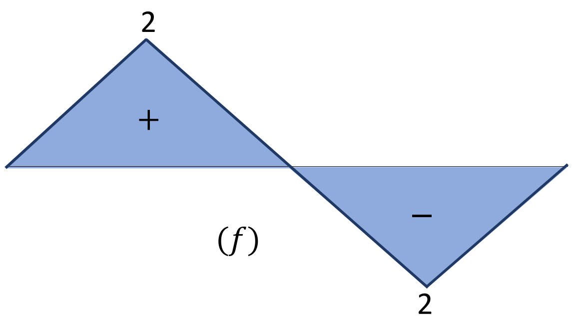


(d)

Influence line for reaction B

(e)

Influence line for reaction D

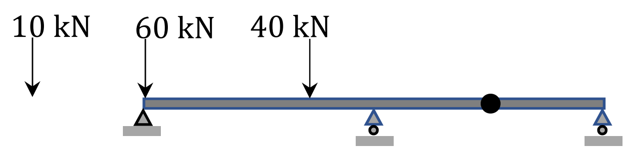


(f)

Influence line for moment M_n

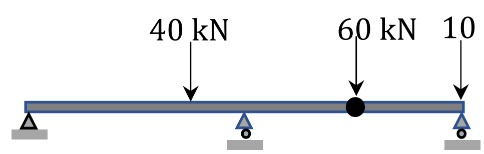
Solution

Maximum positive reaction A_y from Figure 9.21c.



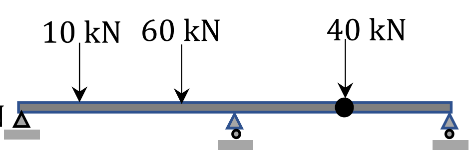
$$A_y = (60)(1) + (40)\left(\tfrac{1}{4}\right) = 70 \text{ kN} \quad \text{Ans}$$

Maximum reaction B_y from Figure 9.21d.



Case 1

$$\text{Case 1: } B = (40)\left(\frac{3}{4}\right) + (60)\left(\frac{3}{2}\right) + (10)(0) = 120 \text{ kN}$$

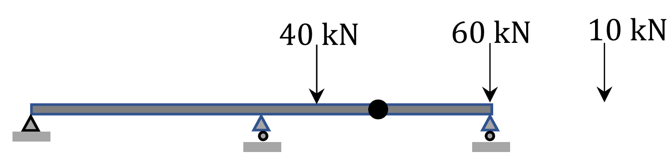


Case 2

$$\text{Case 2: } B = (10)\left(\frac{1}{4}\right) + (60)\left(\frac{3}{4}\right) + (40)\left(\frac{3}{2}\right) = 107.5 \text{ kN}$$

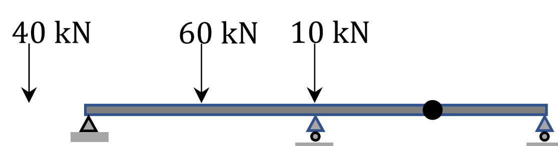
$$\text{Maximum } +B_y = 120 \text{ kN}$$

Maximum positive reactionD_y from Figure 9.21e.



$$D_y = (60)(1) + (40)(0) = 60 \text{ kN}$$

Maximum positive moment M_n from Figure 9.21f.



$$M_n = (60)(2) + (10)(0) = 120 \text{ kN.m}$$

9.4.4 Uses of Influence Lines to Determine Absolute Maximum Response Function at Any Point Along the Structure

The preceding sections explain the use of influence lines for the determination of the maximum response function that may occur at specific points of a structure. This section will explain the determination of the absolute maximum value of a response function that may occur at any point along the entire structure due to concentrated loads exerted by moving loads.

The absolute maximum shear force for a cantilever beam will occur at a point next to the fixed end, while that for a simply supported beam will occur close to one of its reactions. The absolute maximum moment for a cantilever beam will also occur close to the fixed end, while that for simply supported beam is not readily known and, thus, will require some analysis. To locate the position where the absolute maximum moment occurs in a simply supported beam, consider a beam subjected to three moving concentrated loads $P_1, P_2,$ and P_3, as shown in Figure 9.22.

Although it is certain from statics that the absolute maximum moment will occur under one of the concentrated loads, the specific load under which it will occur must be identified, and its location along the beam must be known. The concentrated load under which the absolute maximum moment will occur may be determined by inspection or by trial-and-error process, but the location of this load should be established analytically. Assume that the concentrated load under which the absolute maximum moment will occur is P_3, and the distance of P_3 from the centerline of the beam is x. To obtain an expression for x, first determine the resultant P_R of the concentrated loads, acting at a distance x' from the load P_3.

To determine the right reaction of the beam, take the moment about support *A*, as follows:

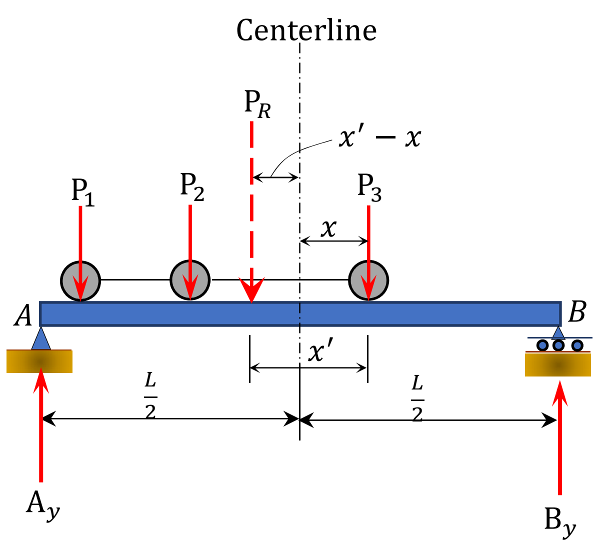


Fig. 9.22. Beam subjected to three moving concentrated loads.

To determine the right reaction of the beam, take the moment about support A, as follows:

$$\sum M_A = 0$$
$$B_y L = P_R\left[\frac{L}{2} - (x' - x)\right]$$

$$B_y = \frac{P_R}{L}\left[\frac{L}{2} - (x' - x)\right] \tag{9.8}$$

Thus, the bending moment under M_3 is as follows:

$$M_3 = B_y\left(\frac{L}{2} - x\right) = \frac{P_R}{L}\left[\frac{L}{2} - (x' - x)\right]\left(\frac{L}{2} - x\right)$$
$$= P_R\left(\frac{L}{4} - + \frac{x'}{2} + \frac{xx'}{L} - \frac{x^2}{L}\right) \tag{9.9}$$

The distance x for which M_3 is maximum can be determined by differentiating equation 9.9 with respect to x and equating it to zero, as follows:

$$\frac{dM_3}{dx} = \mathrm{P}_R\left(\frac{x'}{L} - \frac{2x}{L}\right) = 0$$
$$\frac{2x}{L} = \frac{x'}{L}$$

Therefore,

$$x = \frac{x'}{2} \tag{9.10}$$

Equation 9.10 concludes that the absolute maximum moment in a simply supported beam occurs under one of the concentrated loads when the load under which the moment occurs and the resultant of the system of loads are equidistant from the center of the beam.

Example 9.14

Determine the absolute maximum bending moment in a 16 m-long simply supported girder bridge subjected to a moving truck loading, as shown in Figure 9.23.

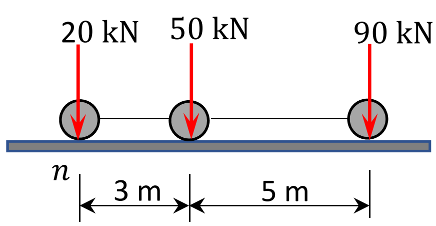


Fig. 9.23. Simply supported girder beam.

Solution

Using statics, first determine the value and the position of the resultant of the moving loads.

Resultant load.

$$P_R = \sum P = 20 + 50 + 90 = 160$$

Position of the resultant load. To determine the position of the resultant load, take the moment about point n, which is directly below the 20 kN load, as follows:

$$\sum M_n: 160x = (50)(3) + (90)(8)$$
$$x = 5.44 \text{ m}$$

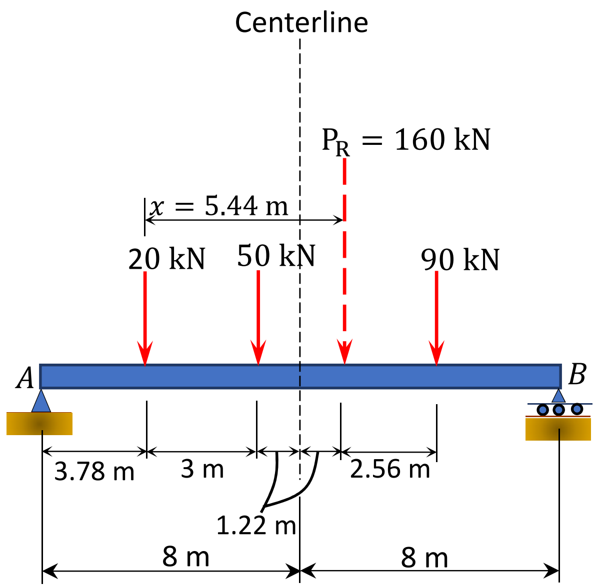


Fig. 9.24. Resultant and load equidistant from centerline of the beam.

If the absolute maximum moment is assumed to occur under the 50 kN load, the positioning of the resultant and this load equidistant from the centerline of the beam is as shown in Figure 9.24. Before computing the absolute maximum moment, first determine the reaction B_y using statics.

$$\sum M_A = 0: -(160)(9.22) + B_y(16) = 0$$
$$B_y = 92.2 \text{ kN}$$

The absolute maximum moment under the 50 kN load is as follows:

$$M_{50} = (92.2)(9.22) - (90)(3.78) = 509.88 \text{ kN.m}$$

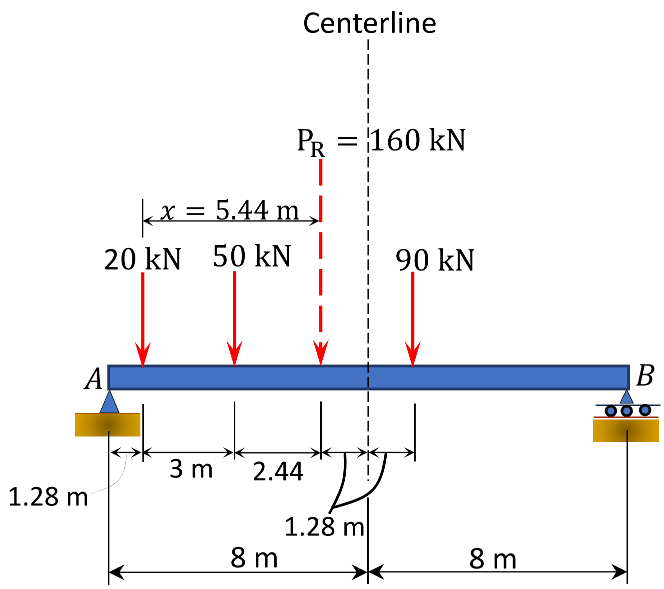


Fig. 9.25. Resultant and load equidistant from centerline of the beam.

If the absolute maximum moment is assumed to occur under the 90 kN load, the positioning of the resultant and this load equidistant from the centerline of the beam will be as shown in Figure 9.25.

Before computing the absolute maximum moment, first determine the reaction B_y using statics.

$$\sum M_A = 0: -(160)(6.72) + B_y(16) = 0$$
$$B_y = 67.2 \text{ kN}$$

The absolute maximum moment under the 90 kN load is as follows:

$$M_{90} = (67.2)(6.72) = 451.58 \text{ kN.m}$$

From the two possible cases considered in the solution, it is evident that the absolute maximum moment occurs under the 50 kN force.

Chapter Summary

Influence lines for statically determinate structures: The effect of a moving load on the magnitude of certain functions of a structure, such as support reactions, deflection, and shear force and moment, at a section of the structure vary with the position of the moving load. Influence lines are used to study the maximum effect of a moving load on these functions for design purposes. The influence lines for determinate structures can be obtained by the static equilibrium method or by the kinematic or Muller-Breslau method. The influence lines by the former method can be determined quantitatively, while those for the latter method can be obtained qualitatively, as have been demonstrated in this chapter. Several example problems are solved showing how to construct the influence lines for beams and trusses using the afore-stated methods.

Practice Problems

9.1 Draw the influence line for the shear force and moment at a section n at the midspan of the simply supported beam shown in Figure P9.1.

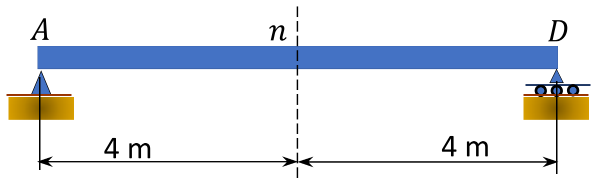


Fig. P9.1. Simply supported beam.

9.2 Draw the influence lines for the reaction at A and B and the shear and the bending moment at point C of the beam with overhanging ends, as shown in Figure P9.2.

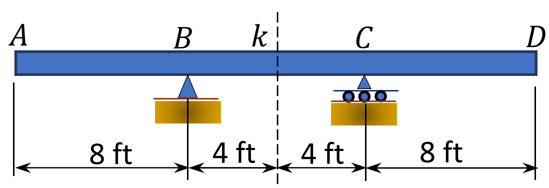


Fig. P9.2. Beam with overhang.

9.3 Draw the influence line for the reactions at the support of the cantilever beam shown in Figure P9.3.

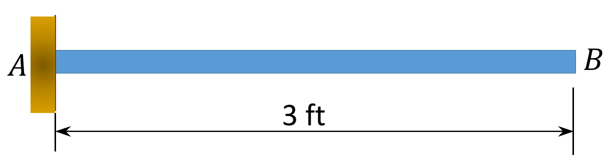


Fig. P9.3. Cantilever beam.

9.4 Draw the influence line for the support reactions at B and D and shear and bending moments at section n of the beam shown in Figure 9.4.

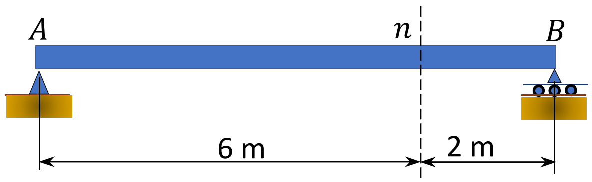


Fig. P9.4. Beam

9.5 Draw the influence lines for support reactions at *C* and *D* and at point *B* of the compound beam shown in Figure P9.5.

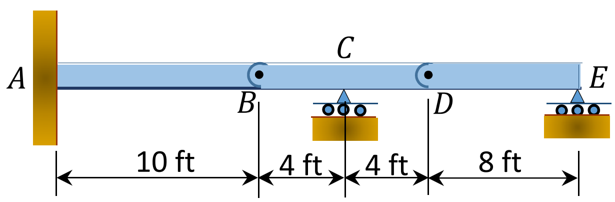


Fig. P9.5. Compound beam.

9.6 Draw the influence lines for the shear force and moment at sections *n*s and *k* of the compound beam shown in Figure P9.6.

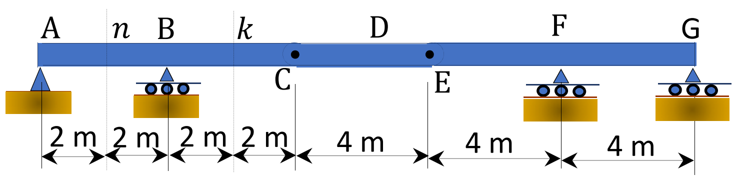


Fig. P9.6. Compound beam.

9.7 Determine the absolute maximum bending moment in a 65 ft-long simply supported girder bridge subjected to a moving truck loading, as shown in Figure P9.7.

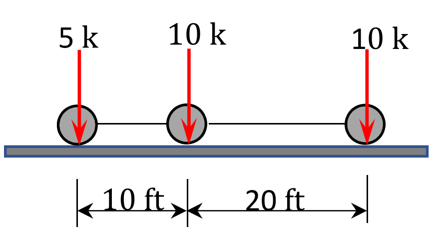


Fig. P9.7. Simply supported girder bridge.

9.8 Determine the absolute maximum bending moment in a 12 m-long simply supported girder bridge subjected to a moving truck loading, as shown in Figure P9.8.

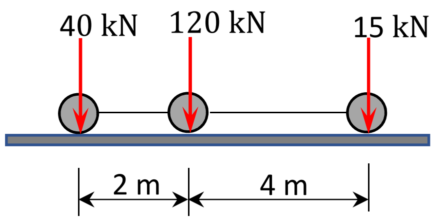


Fig. P9.8. Simply supported girder bridge.

9.9 Determine the absolute maximum bending moment in a 40 ft-long simply supported girder bridge subjected to a moving truck loading, as shown in Figure P9.9.

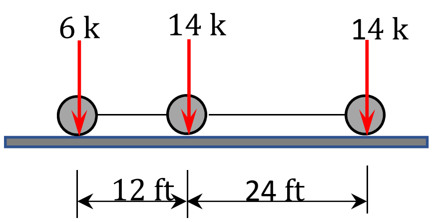


Fig. P9.9. Simply supported girder bridge.

9.10 Determine the absolute maximum bending moment in a 14 m-long simply supported girder bridge subjected to a moving truck loading, as shown in Figure P9.10.

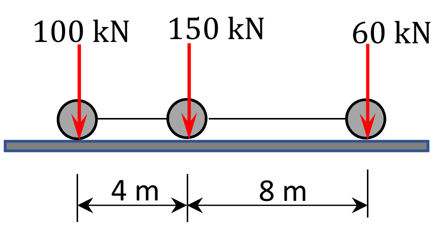


Fig. P9.10. Simply supported girder bridge.

9.11 Draw the influence lines for the moment at *B* and the shear force in panel *CD* of the floor girder shown in Figure P9.11.

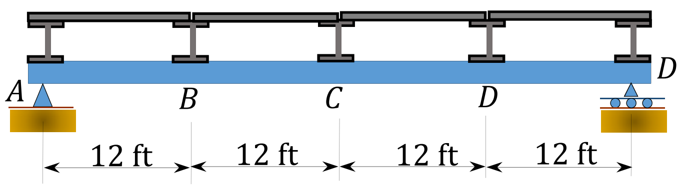


Fig. P9.11. Floor girder.

9.12 Draw the influence lines for the moment at *C* and the shear force in panel *BC* of the floor girder shown in Figure P9.12.

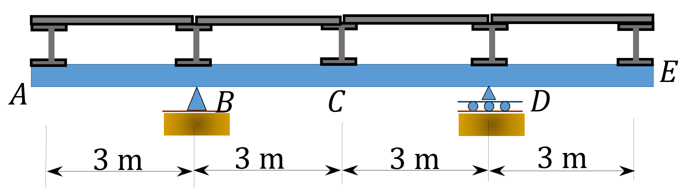


Fig. P9.12. Floor girder.

9.13 Draw the influence lines for the moment at *B* and the shear in panel *CD* of the floor girder shown in Figure P9.13.

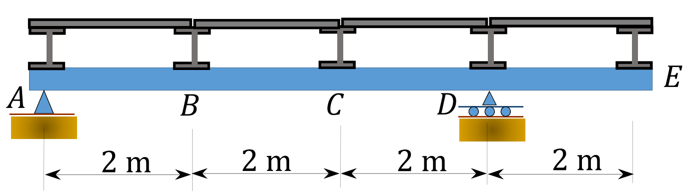


Fig. P9.13. Floor girder.

9.14 Draw the influence lines for the moment at *D* and the shear force in panel *DE* of the floor girder shown in Figure P9.14.

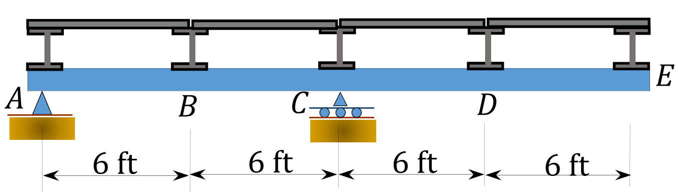


Fig. P9.14. Floor girder.

9.15 Draw the influence lines for the moment at *D* and the shear force in panel *AB* of the floor girder shown in Figure P9.15.

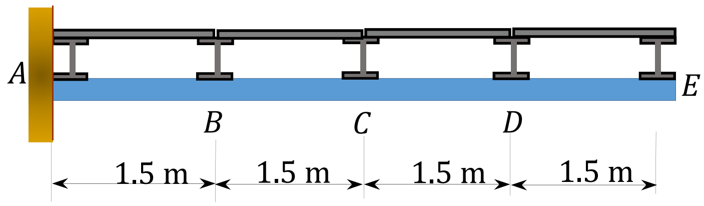


Fig. P9.15. Floor girder.

9.16 Draw the influence lines for the forces in members *CD*, *CF*, and *GF* as a unit load moves across the top of the truss, as shown in Figure P9.16.

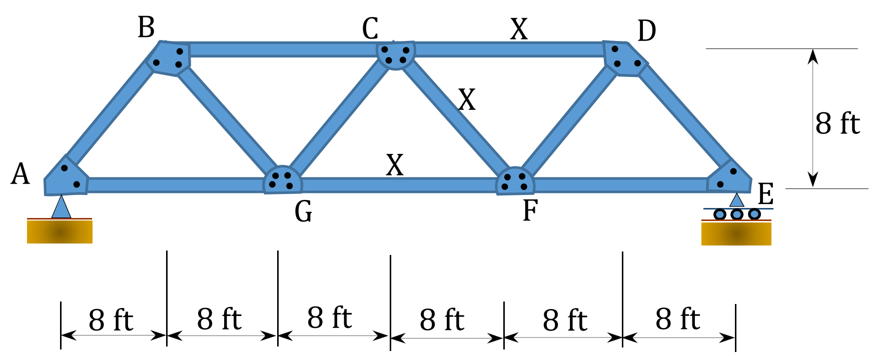


Fig. P9.16. Truss.

9.17 Draw the influence lines for the forces in members *DE*, *NE*, and *NM* as a unit live load is transmitted to the top chords of the truss, as shown in Figure P9.17.

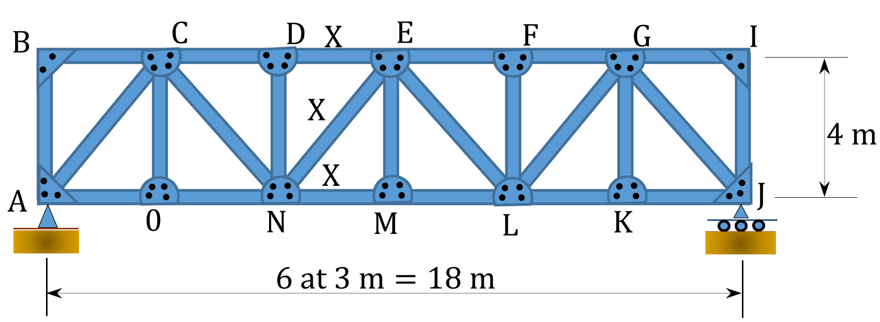


Fig. P9.17. Truss.

9.18 Draw the influence lines for the forces in members *DE*, *DH*, *IH*, and *HG* as a unit live load is transmitted to the bottom chords of the truss, as shown in Figure P9.18.

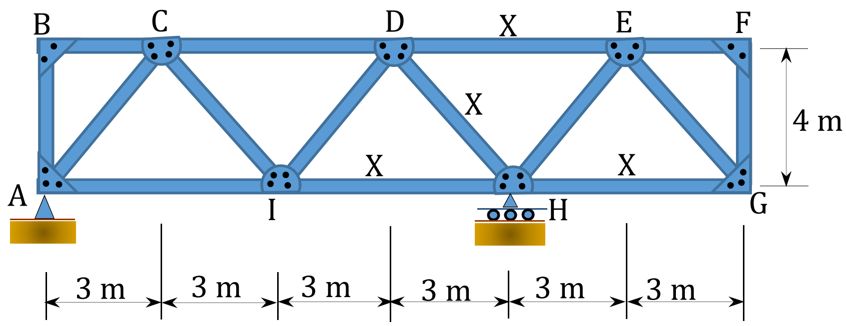


Fig. P9.18. Truss.

9.19 Draw the influence lines for the forces in members *BC*, *BF*, *FE*, and *ED* as a unit load moves across the bottom chords of the truss, as shown in Figure P9.19.

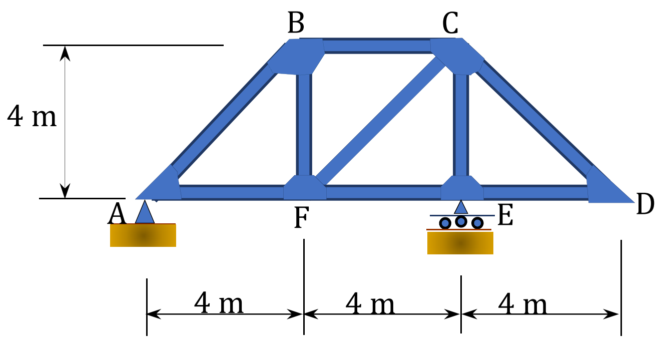


Fig. P9.19. Truss.

PART THREE
ANALYSIS OF STATICALLY INDETERMINATE STRCTURES

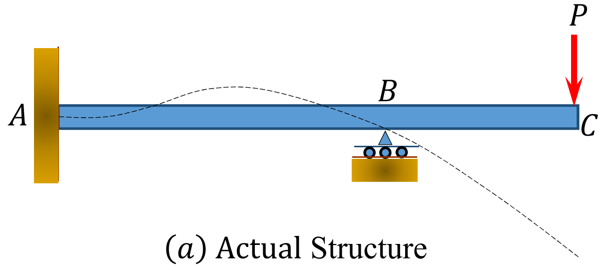
Chapter 10

Force Method of Analysis of Indeterminate Structures

10.1 Introduction

The force method of analysis, also known as the method of consistent deformation, uses equilibrium equations and compatibility conditions to determine the unknowns in statically indeterminate structures. In this method, the unknowns are the redundant forces. A redundant force can be an external support reaction force or an internal member force, which if removed from the structure, will not cause any instability. This method entails formulating a set of compatibility equations, depending on the number of the redundant forces in the structure, and solving these equations simultaneously to determine the magnitude of the redundant forces. Once the redundant forces are known, the structure becomes determinate and can be analyzed completely using the conditions of equilibrium.

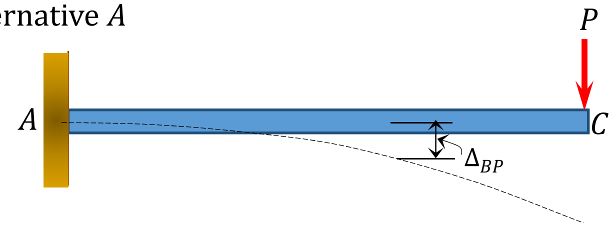
For an illustration of the method of consistent deformation, consider the propped cantilever beam shown in Figure 10.1a. The beam has four unknown reactions, thus is indeterminate to the first degree. This means that there is one reaction force that can be removed without jeopardizing the stability of the structure. The structure that remains after the removal of the redundant reaction is called the primary structure. A primary structure must always meet the equilibrium requirement. A careful observation of the structure being considered will show that there are two possible redundant reactions and two possible primary structures (see Fig. 10b and Fig. 10d). Taking the vertical reaction at support B and the reactive moments at support A as the redundant reactions, the primary structures that remain are in a state of equilibrium. After choosing the redundant forces and establishing the primary structures, the next step is to formulate the compatibility equations for each case by superposition of some sets of partial solutions that satisfy equilibrium requirements. Equations 10.1 and 10.2 satisfy options 1 and 2, respectively. The terms $\Delta_{BP}, \theta_{AP}, \delta_{BB}$, and α_{AA} are referred to as flexibility or compatibility coefficients or constants. The first subscript in a coefficient indicates the position of the displacement, and the second indicates the cause and the direction of the displacement. For example, Δ_{BP} implies displacement at point B caused by the load P in the direction of the load P. The compatibility coefficients can be computed using the Maxwell-Betti Law of Reciprocal, which will be discussed in the subsequent section.



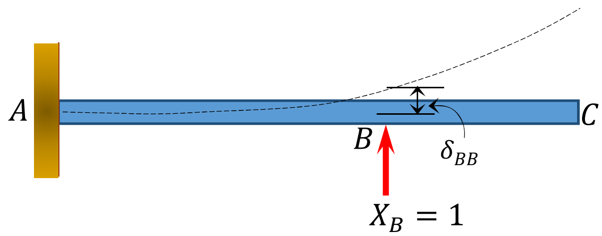
(a) Actual Structure

Fig. 10.1. Propped cantilever beam.

Alternative A

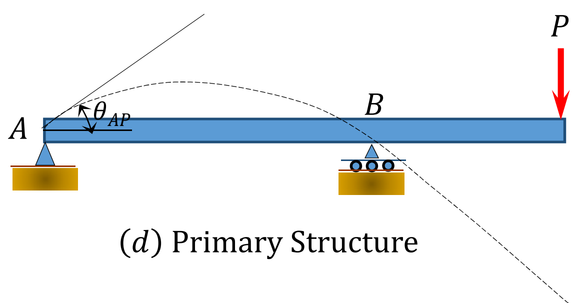


(b) Primary Structure

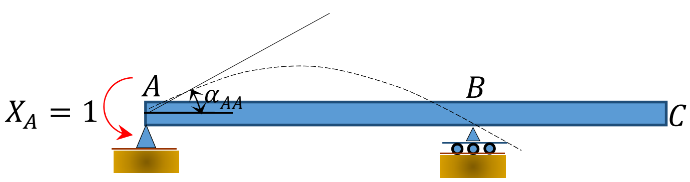


(c) Redundant X_B of unit load applied

Alternative B



(d) Primary Structure



(e) Redundant X_B of unit moment applied

$$\Delta_{BP} + R_B \delta_{BB} = 0 \tag{10.1}$$

$$\theta_{AP} + M_A \alpha_{AA} = 0 \tag{10.2}$$

where

M = moment in the primary structure due to the applied load P.
m = moment in the primary structure due to a unit load applied at B.

m_θ = moment in the primary structure due to a unit moment applied at *A*.

Procedure for Analysis of Indeterminate Structures by the Method of Consistent Deformation

- Determine the degree of indeterminacy of the structure.
- Choose the redundant reactions from the indeterminate structure.
- Remove the chosen redundant reactions to obtain the primary structure.
- Formulate the compatibility equations. The number of the equations must match the number of redundant forces.
- Compute the flexibility coefficients.
- Substitute the flexibility coefficients into the compatibility equations.
- In the case of several redundant reactions, solve the compatibility equations simultaneously to determine the redundant forces or moments.
- Apply the computed redundant forces or moments to the primary structure and evaluate other functions, such as bending moment, shearing force, and deflection, if desired, using equilibrium conditions.

10.2 Maxwell-Betti Law of Reciprocal Deflections

The Maxwell-Betti law of reciprocal deflections establishes the fact that the displacements at two points in an elastic structure subjected to a unit load successively at those points are the same in magnitude. This law helps reduce the computational efforts required to obtain the flexibility coefficients for the compatibility equations when analyzing indeterminate structures with several redundant restraints by force method. The Maxwell-Betti law of reciprocal deflection states that the linear displacement at point *A* due to a unit load applied at *B* is equal in magnitude to the linear displacement at point *B* due to a unit load applied at *A* for a stable elastic structure.

To prove the Maxwell-Betti law of reciprocal deflections, consider a beam subjected to the loads P_1 and P_2 at point 1 and point 2, successively, as shown in Figure 10.2a and Figure 10.2b.

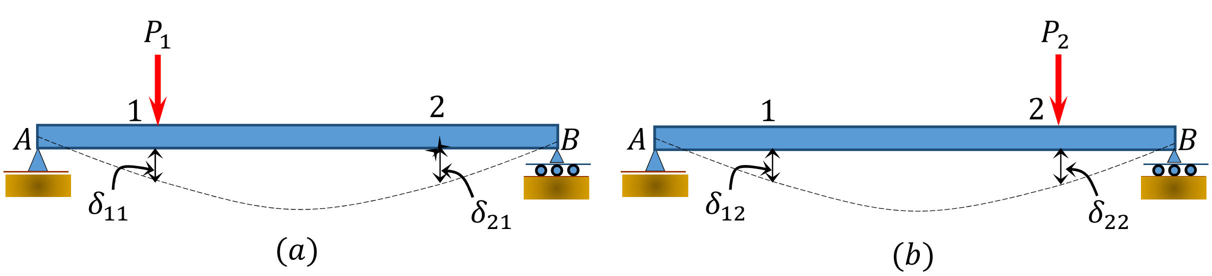


Fig. 10.2. Beam subjected to loads.

Case 1**:**

Apply P_1, followed by P_2.

Work done at point 1 when P_1 is applied:

$$W_1 = \tfrac{1}{2} P_1 \delta_{11} \tag{1}$$

where

δ_{11} = the deflection at point 1 due to the gradually applied load P_1.

Work done at points 1 and 2 when P_2 is applied and P_1 is still in place:

$$W_2 = P_1 \delta_{12} + \tfrac{1}{2} P_2 \delta_{22} \tag{2}$$

where

δ_{12} and δ_{22} = the deflections at point 1 and point 2, respectively, when the load P_2 is gradually at point 2.

Total work done W_T:

$$\begin{aligned} W_T &= W_1 + W_2 \\ &= \tfrac{1}{2} P_1 \delta_{11} + \tfrac{1}{2} P_2 \delta_{22} + P_1 \delta_{12} \end{aligned} \tag{3}$$

Case 2:

Apply P_2, followed by P_1.

Work done at point 1 when P_1 is applied:

$$W_2 = \tfrac{1}{2} P_2 \delta_{22} \tag{4}$$

Work done at points 1 and 2 when P_1 is applied and P_2 is still in place:

$$W_2 = P_2 \delta_{21} + \tfrac{1}{2} P_1 \delta_{11} \tag{5}$$

Total work done W_T:

$$\begin{aligned} W_T &= W_1 + W_2 \\ &= \tfrac{1}{2} P_1 \delta_{11} + \tfrac{1}{2} P_2 \delta_{22} + P_2 \delta_{21} \end{aligned} \tag{6}$$

Equate the total of both cases (from equations 3 and 6).

$\frac{1}{2}P_1\delta_{11} + \frac{1}{2}P_2\delta_{22} + P_1\delta_{12} = \frac{1}{2}P_1\delta_{11} + \frac{1}{2}P_2\delta_{22} + P_2\delta_{21}$

$$P_1\delta_{12} = P_2\delta_{21} \tag{7}$$

Substituting $P_1 = P_2 = 1$ into equation 7 suggests the following:

$$\delta_{12} = \delta_{21} \tag{10.3}$$

The Maxwell-Betti law is also applicable for reciprocal rotation. The theorem for reciprocal rotation states that the rotation at point *B* due to a unit couple moment applied at point *A* is equal in magnitude to the rotation at *A* due to a unit couple moment applied at point *B*. This is expressed as follows:

$$\alpha_{AB} = \alpha_{BA} \tag{10.4}$$

where

α_{AB} = the rotation at a point *A* due to a unit couple moment applied at *B*.
α_{BA} = the rotation at a point *B* due to a unit couple moment applied at *A*.

10.3 Analysis of Indeterminate Beams and Frames

The analyses of indeterminate beams and frames follow the general procedure described previously. First, the primary structures and the redundant unknowns are selected, then the compatibility equations are formulated, depending on the number of the unknowns, and solved. There are several methods of computation of flexibility coefficients when analyzing indeterminate beams and frames. These methods include the use of the Mohr integral, deflection tables, and the graph multiplication method. These methods are illustrated in the solved example problems in this section.

10.3.1 Computation of Flexibility Coefficients Using the Mohr Integral

The Mohr integral for obtaining the flexibility coefficient for beams and frames is expressed as follows:

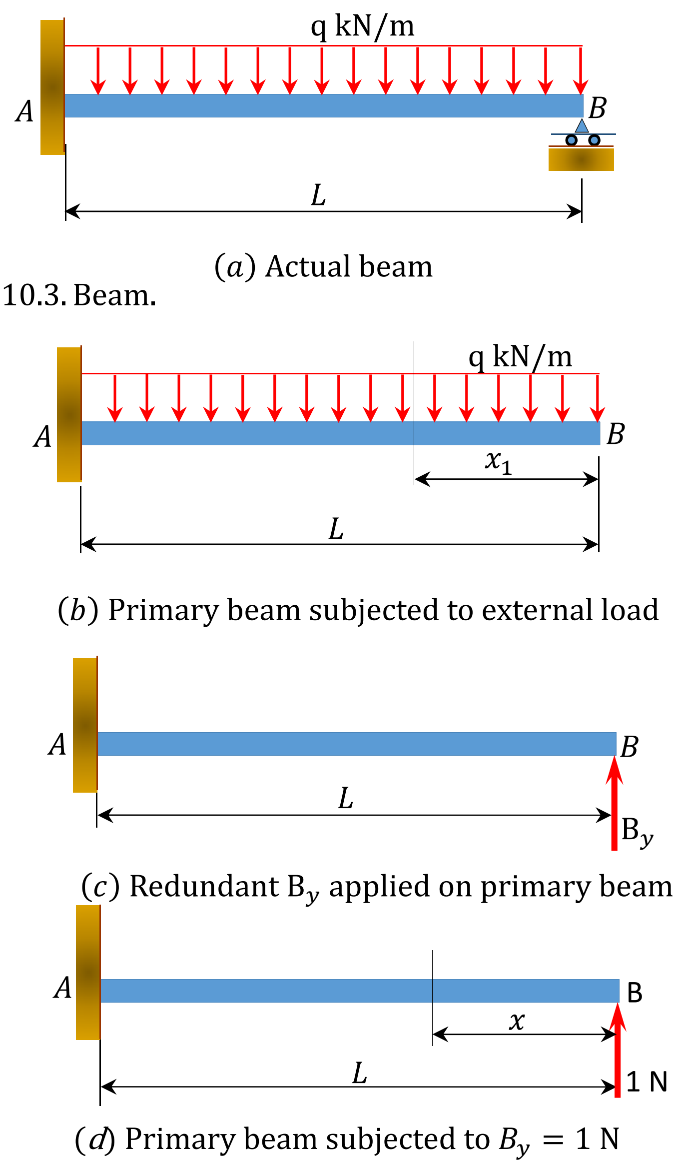
$$\begin{aligned} \Delta_{BP} &= \int \frac{Mm}{EI}dx \\ \delta_{BB} &= \int \frac{m^2}{EI}dx \\ \theta_{AP} &= \int \frac{Mm_\theta}{EI}dx \\ \alpha_{AA} &= \int \frac{m_\theta^2}{EI}dx \end{aligned} \tag{10.5}$$

where

M = moment in the primary structure due to the applied load P.
m = moment in the primary structure due to a unit load applied at B.
m_θ = moment in the primary structure due to a unit moment applied at A.

Example 10.1

Determine the reactions in the beam shown in Figure 10.3a. Use the method of consistent deformation to carry out the analysis. All flexibility coefficients are determined by integration. EI = constant.



(a) Actual beam

Fig. 10.3. Beam.

(b) Primary beam subjected to external load

(c) Redundant B_y applied on primary beam

(d) Primary beam subjected to $B_y = 1$ N

Solution

Classification of structure. There are four unknown reactions in the beam: three unknown reactions at the fixed end A and one unknown reaction at the prop B. Since there are three equations of equilibrium on a plane, it implies that the beam has one unknown reaction in excess of the equations of equilibrium on a plane, thus it is indeterminate to one degree.

Choice of primary structure. There may be more than one possible choice of primary structure. For the given propped cantilever beam, the prop at B will be selected as the redundant. Thus, the primary structure is as shown in Figure 10.3b.

Compatibility equation. The number of compatibility equations will always match the number of the redundant reactions in a given structure. For the given cantilever beam, the number of compatibility equations is one and is written as follows:

$$\Delta_{BP} + B_y \delta_{BB} = 0$$

The flexibility or compatibility coefficients Δ_{BP} and δ_{BB} can be computed by several methods, including the integration method, the graph multiplication method, and the table methods. For this example, the flexibility coefficients are computed using the integration method.

The bending moment expressions for the primary beam subjected to external loading is written as follows:

$0 < x < L$

$$M = -\frac{qx^2}{2}$$

The bending moment in the primary beam subjected to $B_y = 1$ kN is as follows:

$$M = x$$

$$\Delta_B = \Delta_{BP} + R_B \delta_{BB} = 0$$

Using integration to obtain the flexibility coefficients suggests the following:

$$\Delta_{BP} = \int \frac{Mm}{EI} dx$$

$$= \int_0^L \left(\frac{-qx^2}{2}\right)(x)\frac{dx}{EI} = -\frac{qL^4}{8EI}$$

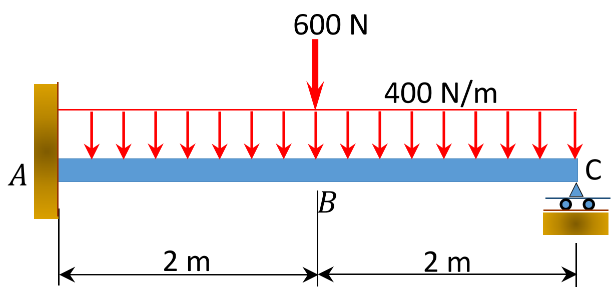
$$\delta_{BB} = \int_0^L \frac{m^2}{EI} dx = \int_0^L \frac{x^2}{EI} dx = \frac{L^3}{3EI}$$

Putting the computed flexibility coefficients into the compatibility equation suggests the following:

$$B_y = -\frac{\Delta_{BP}}{\delta_{BB}} = -\left(\frac{-qL^4}{8EI}\right)\left(\frac{3EI}{L^3}\right) = \frac{3qL}{8}$$

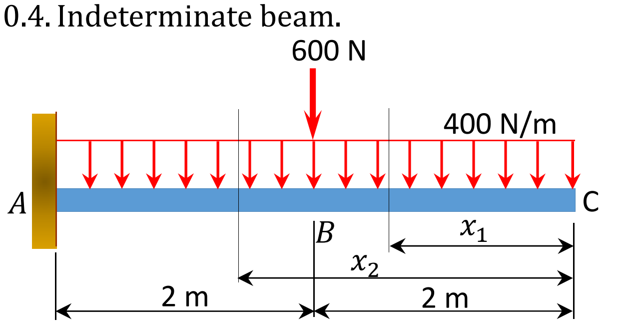
Example 10.2

Determine the support reactions and draw the bending moment and the shearing force diagrams for the indeterminate beam shown in Figure 10.4. Use the method of consistent deformation. EI = constant.

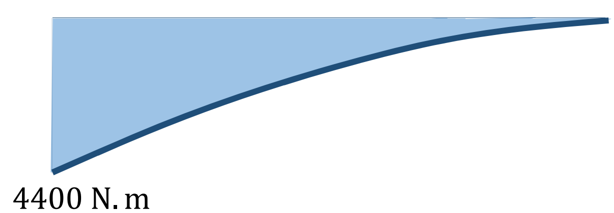


(a) Actual beam

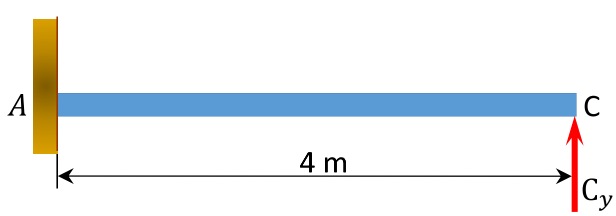
Fig. 10.4. Indeterminate beam.



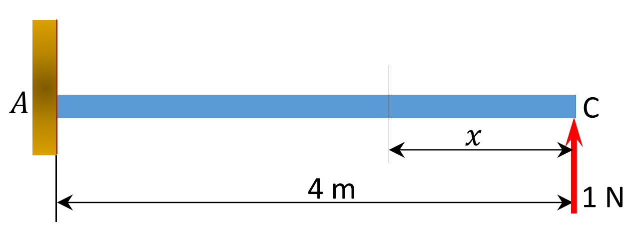
(b) Primary beam subjected to external load



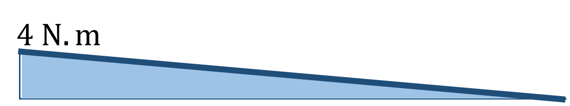
(f) Bending moment diagram for primary beam due to external loading, M_p



(c) Redundant C_y applied on primary beam



(d) Primary beam subjected to $C_y = 1$ N



(g) Bending moment diagram for primary beam due to $C_y = 1$, m

Solution

Classification of structure. There are four unknown reactions in the beam: three unknown reactions at the fixed end *A* and one unknown reaction at the prop *C*. Since there are three equations of equilibrium on a plane, it implies that the beam has one unknown reaction in excess of the equations of equilibrium on a plane. Thus, it is indeterminate to one degree.

Choice of primary structure. There may be more than one possible choice of primary structure. For the given propped cantilever beam, the reaction at *C* is selected as the redundant reaction. Thus, the primary structure is as shown in Figure 10.4b.

Compatibility equation. The number of compatibility equations will always match the number of the redundant reactions in a given structure. For the given cantilever beam, the number of compatibility equations is one and is written as follows:

$$\Delta_{CP} + C_y\delta_{CC} = 0$$

The flexibility or compatibility coefficients Δ_{CP} and δ_{CC} are computed using the integration method.

The bending moment expressions for segments *AB* and *BC* of the primary beam subjected to an external loading is written as follows:

$0 < x_1 < 2$

$M = -\frac{400x^2}{2} = -200x^2$

$2 < x_2 < 4$

$M = -\frac{400x^2}{2} - 600(x-2) = -200x^2 - 600(x-2)$

The bending moment in the primary beam subjected to $C_y = 1\text{N}$ is written as follows:

$M = x$

$$\Delta_{1P} = \int_0^2 \frac{mM_p dx}{EI} + \int_2^4 \frac{mM_p dx}{EI}$$

$$\Delta_{1P} = \int_0^2 \frac{(x)(-200x^2)dx}{EI} + \int_2^4 \frac{(x)[-200x^2 - 600(x-2)]dx}{EI}$$

$$= -\frac{16800}{EI}$$

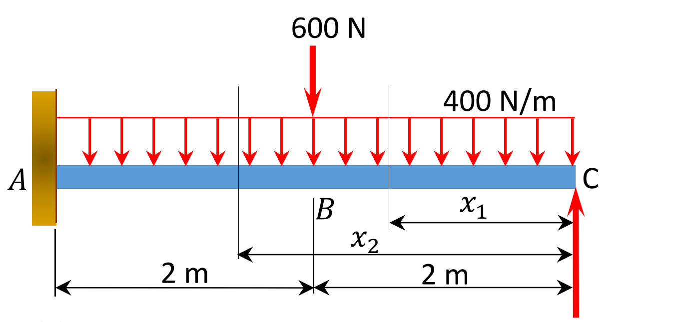
$$\Delta\delta_{c1} = \int_0^4 \frac{(x^2)dx}{EI}$$

$$= \frac{21.33}{EI}$$

Putting the computed flexibility coefficients into the compatibility equation suggests the following:

$$C_y = -\frac{\Delta_{CP}}{\delta_{c1}} = \frac{16800}{21.33} = 787.63 \text{ N}$$

Shearing force and bending moment diagram. To determine the magnitudes of the shearing force and the bending moment and draw their diagrams, apply the obtained redundant to the primary beam, as shown in Figure 10.4e.



(e)

$0 < x_1 < 2$

$V = -787.63 + 400x$

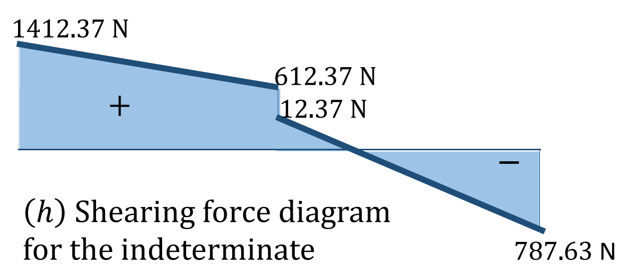
When $x = 0, V = -787.63$ N

When $x = 2, V = 12.37$ N

$M = 787.63x - \frac{400x^2}{2}$

When $x = 0, M = 0$

When $x = 2, M = 775.26$ N.m



(h) Shearing force diagram for the indeterminate beam

$2 < x_2 < 4$

$V = -787.63 + 400x + 600$

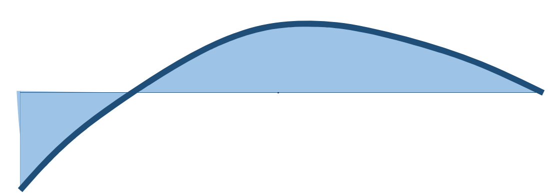
When $x = 2$ m, $V = 612.37$ N

When $x = 4$ m, $V = 1412.37$ N

$M = 787.63x - \frac{400x^2}{2} - 600(x-2)$

When $x = 2$ m, $M = 775.26$ N.m

When $x = 4$ m, $M = -1249.48$ N.m



(i) Bending moment diagram for the indeterminate beam

The shearing force and the bending moment diagrams are shown in Figure 10.4h and Figure 10.4i.

10.3.2 Computation of Flexibility Coefficients by Graph Multiplication Method

The computation of the flexibility coefficients for the compatibility equations by the method of integration can be very lengthy and cumbersome, especially for indeterminate structures with several unknown redundant forces. In such instances, obtaining the coefficients by the graph multiplication method is time-saving. The graph multiplication method is based on the premise that the integral $\int \frac{Mm}{EI} dx$ contains the product of two moment graphs M and m. To derive the formula for the graph multiplication method, consider the two moment diagrams M' and M, as shown in Figure 10.5. The graph of M' is linear, while that of M is of an arbitrary function.

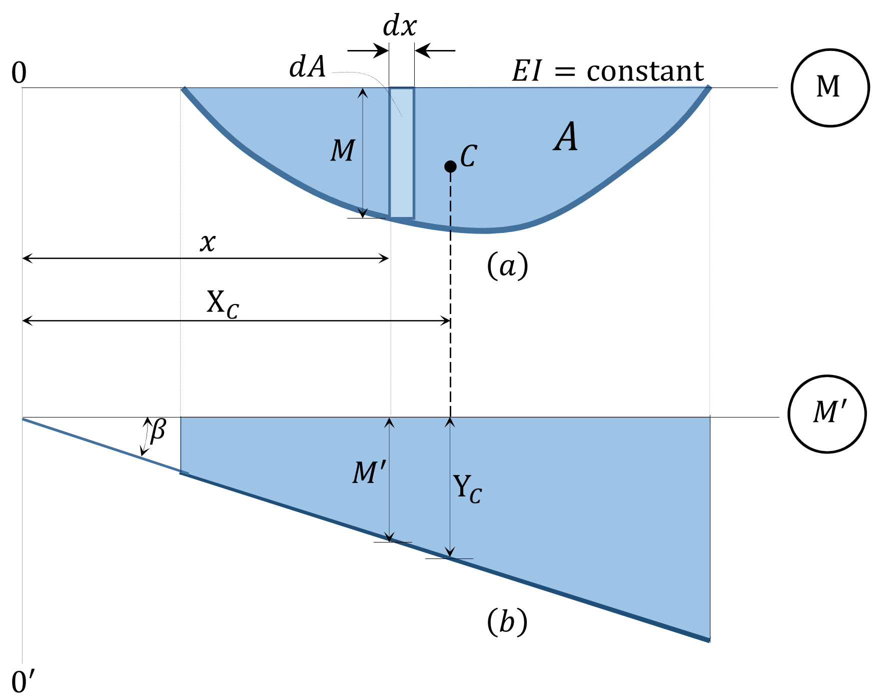


Fig. 10.5. Moment diagrams.

Assuming the flexural rigidity EI is constant, the integral of the product of these two moment diagrams can be expressed as follows:

$$\int \frac{MM'}{EI} dx \tag{1}$$

The elementary area of the bending moment diagram at a distance x from the left end, as shown in Figure 10.5a, is written as follows:

$$dA = Mdx \tag{2}$$

Using trigonometry, the ordinate M' of the linear graph M' at a distance x from the origin, as shown in Figure 10.5b, can be expressed as follows:

$$Y_c = x.tan\beta \quad (3)$$

Substituting equation 2 and 3 into equation 1 suggests the following:

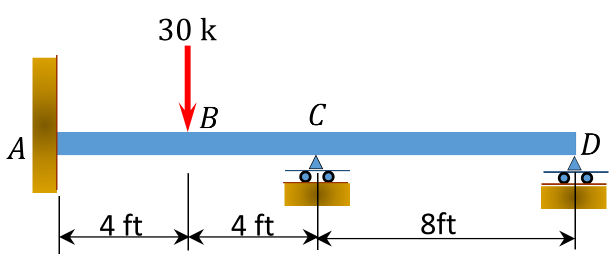
$$\begin{aligned} \int \frac{Mm}{EI} dx &= \int dA.x.tan\varphi \\ &= tan\beta \int dA.x \\ &= xtan\beta.A \\ &= AY_c \end{aligned}$$

$$\int \frac{MM}{EI} dx = AY_c \quad (10.6)$$

As suggested by equation 10.6, the integral of the product of two moment diagrams is equal to the product of the area of one of the moment diagrams (preferably the diagram with the arbitrary outline) and the ordinate in the second moment diagram with a straight outline, lying on a vertical line passing through the centroid of the first moment diagram.

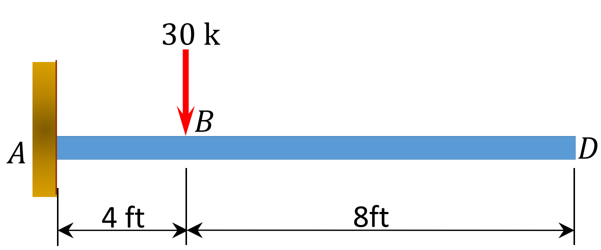
Example 10.3

Determine the reactions at supports A, C, and D of the beam shown in Figure 10.6a. A is a fixed support, while C and D are roller supports. EI = constant.

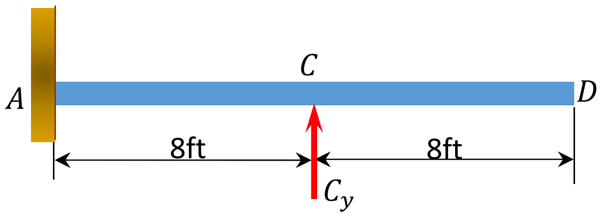


(a) Actual beam

Fig. 10.6. Beam.

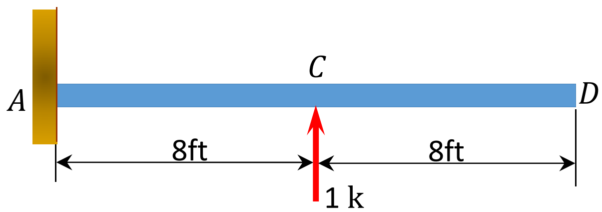


(b) Primary beam subjected to external load

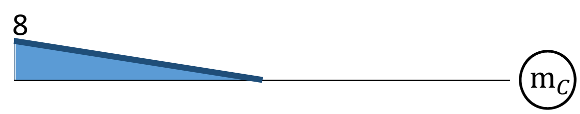


(c) Redundant C_y applied on primary beam

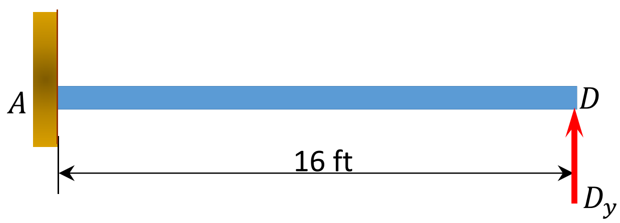
(g) Bending moment diagram for primary beam due to external



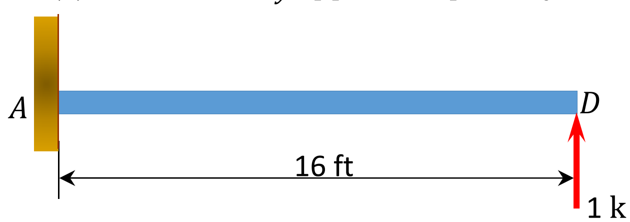
(d) Primary beam subjected to $C_y = 1$



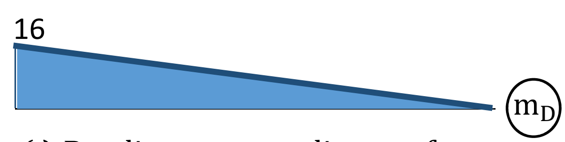
(h) Bending moment diagram for primary beam due to $C_y = 1$ k



(e) Redundant D_y applied on primary beam



(f) Primary beam subjected to $D_y = 1$



(i) Bending moment diagram for primary beam due to $D_y = 1$ k

Solution

Classification of structure. There are five unknown reactions in the beam. Thus, the degree of indeterminacy of the structure is two.

Choice of primary structure. The supports at C and D are chosen as the redundant reactions. Therefore, the primary structure is a cantilever beam subjected to the given concentrated load shown in Figure 10.6b. The primary structure subjected to the redundant unknowns are shown in Figure 10.6c, Figure 10.6d, Figure 10.6e, and Figure 10.6f.

Compatibility equation. There are two compatibility equations, as there are two redundant unknown reactions. The equations are as follows:

$$\Delta_{CP} + C_y\delta_{CC} + D_y\delta_{CD} = 0$$

$$\Delta_{DP} + C_y\delta_{DC} + D_y\delta_{DD} = 0$$

The first alphabets of the subscript of the flexibility coefficients indicate the location of the deflection, while the second alphabets indicate the force causing the deflection. Using the graph multiplication method, the coefficients are computed as follows:

Using the graph multiplication method, the flexibility coefficients are computed as follows:

$$\Delta_{CP} = \left(-\frac{1}{2} \times 4 \times 120\right)(6.67) = -1600.8$$

$$\Delta_{DP} = \left(-\frac{1}{2} \times 4 \times 120\right)(10.67) = -2560$$

$$\delta_{CC} = \left(\frac{1}{2} \times 8 \times 8\right)(5.33) = 170.56$$

$$\delta_{CD} = \delta_{DC} = \left(\frac{1}{2} \times 8 \times 8\right)(13.33) = 426.56$$

$$\delta_{DD} = \left(\frac{1}{2} \times 16 \times 16\right)(10.67) = 1365.76$$

Substituting the flexibility coefficients into the compatibility equation suggests the following two equations, with two unknowns:

$$-1600.8 + 170.56C_y + 426.56D_y = 0$$
$$-2560 + 426.56C_y + 1365.76D_y = 0$$

Solving both equations simultaneously suggests the following:

$C_y = 21.46$ k

$D_y = -4.83$ k

The determination of the reactions at support *A* is as follows:

$$+\curvearrowleft \sum M_A = 0: -(30)(4) - (4.83)(16) + (21.46)(8) + M_A = 0$$

$$M_A = 25.6 \text{ k.ft}$$

$$+\uparrow \sum F_y = 0: A_y - 30 + 21.46 - 4.83 = 0$$

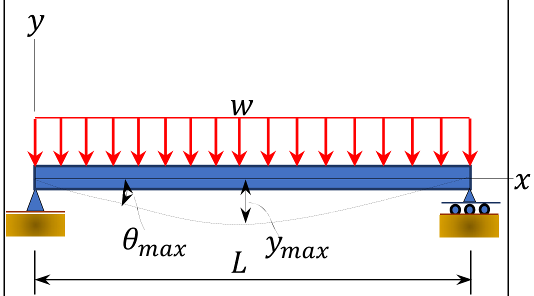
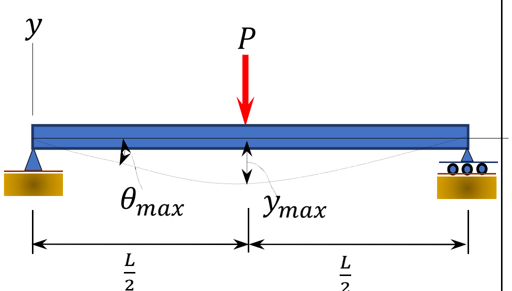
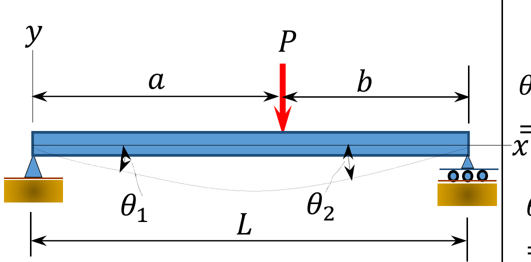
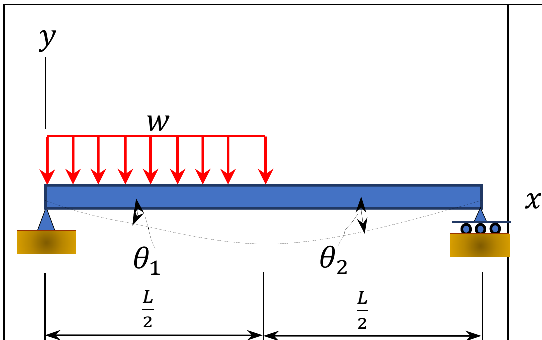
$A_y = 13.37$ k

$$+\rightarrow \sum F_x = 0: A_x = 0$$

10.3.3 Use of Beam-Deflection Tables for Computation of Flexibility Coefficients

This is the easiest method of computation of flexibility coefficients. It involves obtaining the constants from tabulated deflections based on the types of supports and loading configurations, as shown in Table 10.1 and Table 10.2.

Table 10.1. Simply supported beam slopes and deflections.

Beam	Slope	Deflection	Elastic curve
	$\theta_{max} = \frac{-wL^3}{24EI}$	$y_{max} = \frac{-5wL^4}{384EI}$	$y = \frac{-wx}{24EI}(x^3 - 3Lx^2 + L^3)$
	$\theta_{max} = \frac{-PL^2}{16EI}$	$y_{max} = \frac{-PL^3}{48EI}$	$0 \leq x \leq \frac{L}{2}$ $y = \frac{-Px}{48EI}(3L^2 - 4x^2)$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	At $x = a$ $y = \frac{-Pba}{6EI}(L^2 - b^2 - a^2)$	$0 \leq x \leq a$ $y = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	At $x = \frac{L}{2}$ $y = \frac{-5wL^4}{768EI}$ v_{max} is at $x = 0.4598L$ $V_{max} = -0.006563\frac{wL^4}{EI}$	$0 \leq x \leq \frac{L}{2}$ $y = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $\frac{L}{2} \leq x < L$ $y = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$

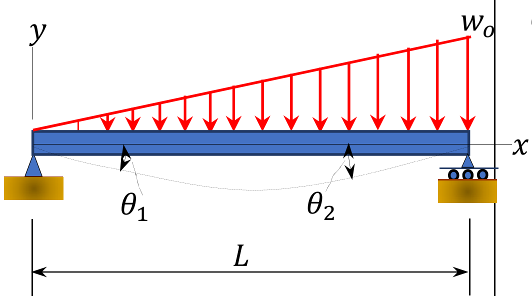
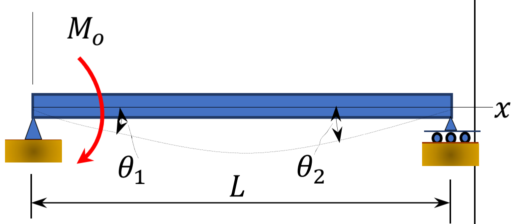
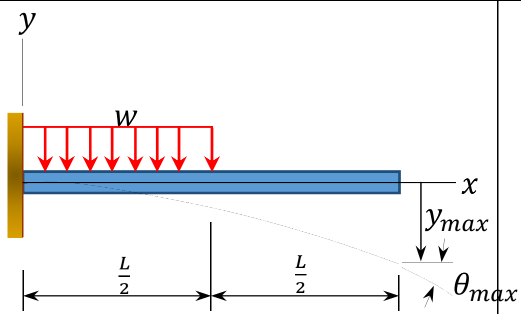
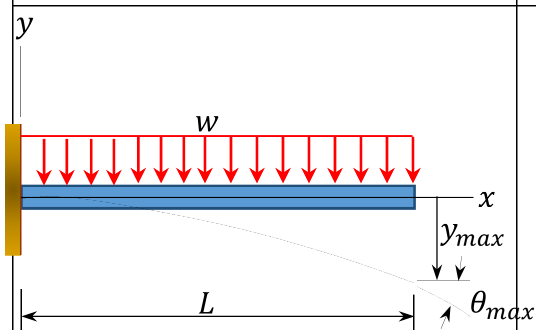
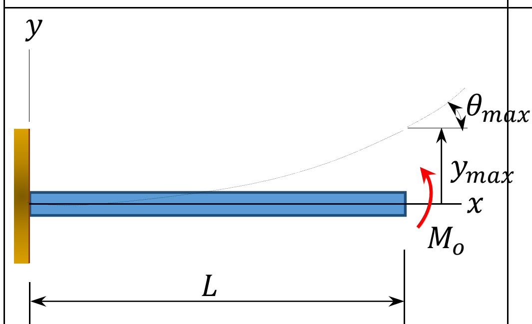
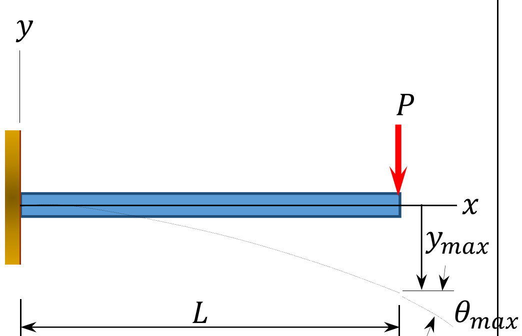
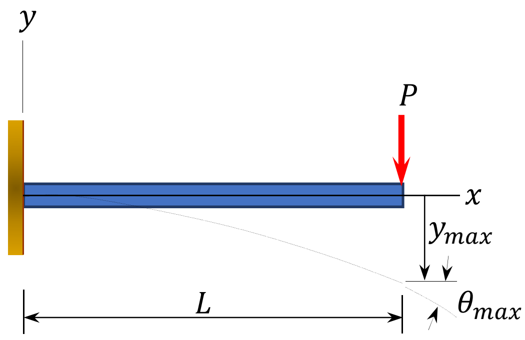
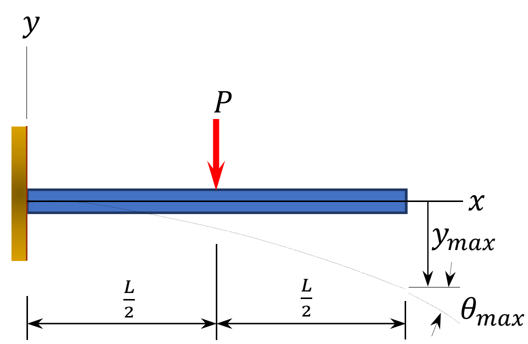
	$\theta_1 = \frac{-7w_oL^3}{360EI}$ $\theta_2 = \frac{w_oL^3}{45EI}$	$v_{max}\ is\ at\ x = 0.5193$ $V_{max} = -0.00652\frac{w_oL^4}{EI}$	$y = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$
	$\theta_1 = \frac{-M_oL}{3EI}$ $\theta_2 = \frac{M_oL}{6EI}$	$y_{max} = \frac{-M_oL^2}{\sqrt{243}EI}$	$y = \frac{-M_ox}{6EIL}(x^2 - 3Lx + 2L^2)$

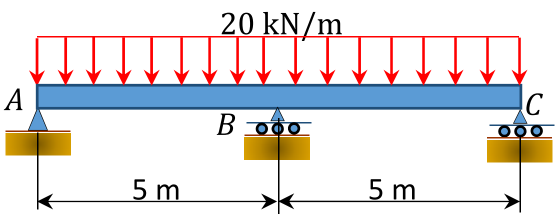
Table 10.2. Cantilevered beam slopes and deflections.

Beam	Slope	Deflection	Elastic curve
	$\theta_{max} = \frac{-wL^3}{48EI}$	$y_{max} = \frac{-7wL^4}{384EI}$	$0 \le x \le \frac{L}{2}$ $y = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $\frac{L}{2} \le x \le L$ $y = \frac{-wL^3}{192EI}\left(4x - \frac{L}{2}\right)$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$y_{max} = \frac{-wL^4}{8EI}$	$y = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \frac{M_oL}{EI}$	$y_{max} = \frac{M_oL^2}{2EI}$	$y = \frac{M_ox^2}{2EI}$
	$\theta_{max} = \frac{-PL^2}{2EI}$	$y_{max} = \frac{-PL^3}{3EI}$	$y = \frac{-Px^2}{6EI}(3L - x)$

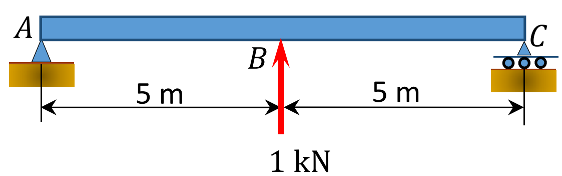
	$\theta_{max} = \frac{-PL^2}{2EI}$	$y_{max} = \frac{-PL^3}{3EI}$	$y = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-PL^2}{8EI}$	$y_{max} = \frac{-5PL^3}{48EI}$	$0 \le x \le \frac{L}{2}$ $y = \frac{-Px^2}{6EI}\left(\frac{3}{2}L - x\right)$ $\frac{L}{2} \le x \le L$ $y = \frac{-PL^2}{24EI}\left(3x - \frac{L}{2}\right)$

Example 10.4

Draw the bending moment and the shearing force for the indeterminate beam shown in Figure 10.7a. EI = constant.

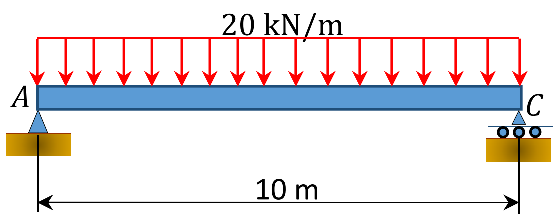


(a) Actual beam

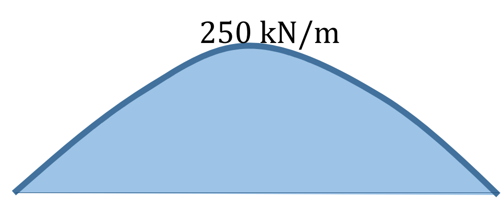


(d)Primary beam subjected to $C_y = 1\text{kN}$

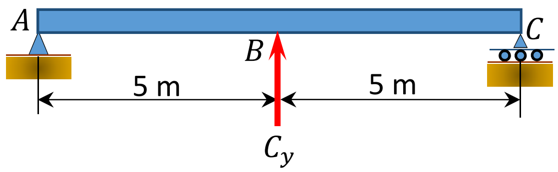
Fig. 10.7. Indeterminate beam.



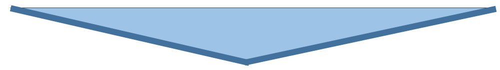
(b) Primary beam subjected to external to external loading



(e) Bending moment diagram for primary beam due to external loading



(c)Redundant C_y applied on primary beam



(f) Bending moment diagram for primary beam due to $B_y = 1$ kN

Solution

Classification of structure. There are four unknown reactions in the beam. Thus, the beam is indeterminate to one degree.

Choice of primary structure. The reaction at B is chosen as the redundant reaction. Thus, the primary structure is a simply supported beam, as shown in Figure 10.7b. Shown in Figure 10.7c and Figure 10.7d are the primary structures loaded with the redundant reactions.

Compatibility equation. The compatibility equation for the beam is written as follows:

$$\Delta_{BP} + B_y \delta_{\boldsymbol{BB}} = 0$$

To compute the flexibility coefficients Δ_{BP} and δ_{BB}, use the beam-deflection formulas in Table 10.1.

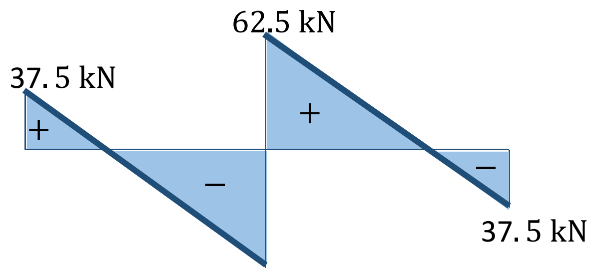
$$\Delta_{BP} = -\frac{5\omega L^4}{384EI} = -\frac{5(20)(10)^4}{384EI} = -\frac{2604.17}{EI}$$

$$\delta_{\boldsymbol{BB}} = \frac{PL^3}{48EI} = \frac{(10)^3}{48EI} = \frac{20.83}{EI}$$

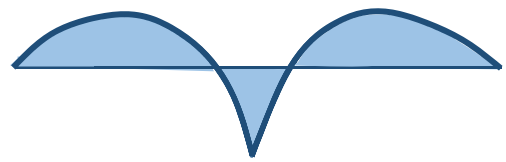
Putting the computed flexibility coefficients into the compatibility equation suggests the following:

$$B_y = \frac{\Delta_{BP}}{\delta_{\boldsymbol{BB}}} = \frac{2604.17}{20.83} = 125 \text{ kN}$$

Shearing force and bending moment diagrams. Once the magnitudes of the redundant reactions are known, the beam becomes determinate and the bending moment and shearing force diagrams are drawn, as shown in Figure 10.7g and Figure 10.7h.



(g) Shearing force diagram for the indeterminate beam



(h) Bending moment diagram for the indeterminate beam.

Example 10.5

To obtain the flexibility coefficients, use the beam-deflection tables to determine the support reactions of the beams in examples 10.1 and 10.2.

Solution

Classification of structure. The degree of indeterminacy of the beam in examples 10.1 and 10.2 is 2.

Flexibility coefficients. Using the information in Table 10.2, determine the flexibility coefficients for example 10.1, as follows:

$$\Delta_{BP} = -\frac{PL^4}{8EI} = -\frac{qL^4}{8EI}$$

$$\delta_{BB} = \frac{PL^3}{3EI} = \frac{qL^3}{3EI}$$

$$B_y = -\frac{\Delta_{BP}}{\delta_{BB}} = -\left(-\frac{qL^4}{8EI}\right)\left(\frac{3EI}{L^3}\right) = \frac{3qL}{8}$$

Using the beam-deflection formulas, obtain the following flexibility coefficients for the beam in example 10.2, as follows:

$$\Delta_{CP} = \frac{-wL^4}{8EI} + \frac{-5PL^3}{48EI} = \frac{-400(4)^4}{8EI} + \frac{-5(600)(4)^3}{48EI} = -\frac{16800}{EI}$$

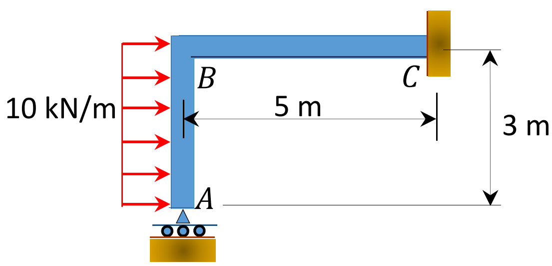
$$\delta_{CC} = \frac{PL^3}{3EI} = \frac{(1)(4)^3}{3EI} = \frac{21.33}{EI}$$

Putting the computed flexibility coefficients into the compatibility equation suggests the following answer:

$$C_y = -\frac{\Delta_{CP}}{\delta_{CC}} = \frac{16800}{21.33} = 787.63 \text{ N}$$

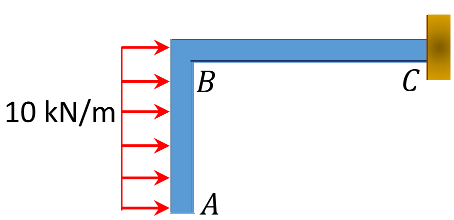
Example 10.6

Using the method of consistent deformation, draw the shearing force and the bending moment diagrams of the frame shown in Figure 10.8a. *EI* = constant.

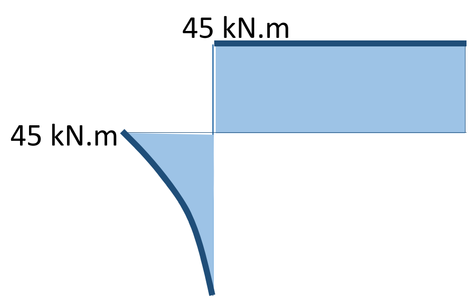


(a) Actual frame

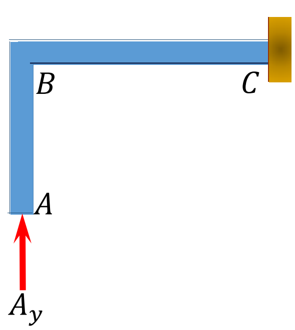
Fig. 10.8. Frame.



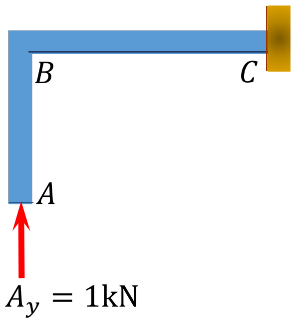
(b) Primary frame subjected to external load



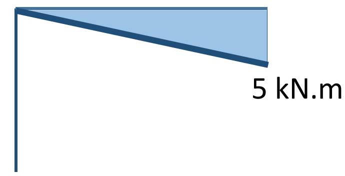
(e) Bending moment for primary beam due to external loading



(c) Redundant A_y applied on Primary frame



(d) Primary frame subjected to $A_y = 1\text{kN}$



(f) Bending moment for primary beam due to $A_y = 1$ kN

Solution

Classification of structure. There are four unknown reactions in the frame: one unknown reaction at the free end A and three unknown reactions at the fixed end C. Thus, the degree of indeterminacy of the structure is one.

Choice of primary structure. Selecting the reaction at support A as the redundant unknown force suggests that the primary structure is as shown in Figure 10.8b. The primary structure loaded with the redundant force is shown Figure 10.8c and Figure 10.8d.

Compatibility equation. The compatibility equation for the indeterminate frame is as follows:

$$\Delta_{AP} + A_y \delta_{AA} = 0$$

The flexibility or compatibility coefficients Δ_{AP} and δ_{AA} are computed by graph multiplication method, as follows:

$$\Delta_{BP} = -\frac{1}{2}(5 \times 5)(45) = -562.5$$

$$\delta_{AA} = \frac{1}{2}(5 \times 5)\left(\frac{10}{3}\right) = 41.67$$

Substituting the flexibility coefficients into the compatibility equation and solving it to obtain the redundant reaction suggests the following:

$$-562.5 + 41.67 A_y = 0$$
$$A_y = 13.5 \text{ kN}$$

Determining the reactions at C.

$$\sum M_C = 0: -(13.5)(5) + (10 \times 3)(1.5) + M_C = 0$$

$$M_C = 22.5.6 \text{ kN.m}$$

$$\sum F_y = 0: -C_y + 13.5 = 0$$

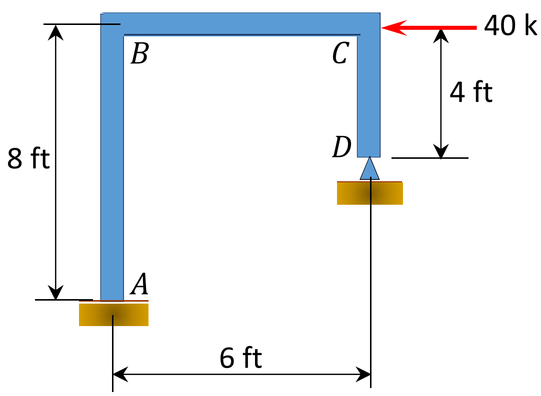
$$C_y = 13.5 \text{ kN}$$

$$\sum F_x = 0: -C_x + (10 \times 3) = 0$$

$$C_x = 30 \text{ kN}$$

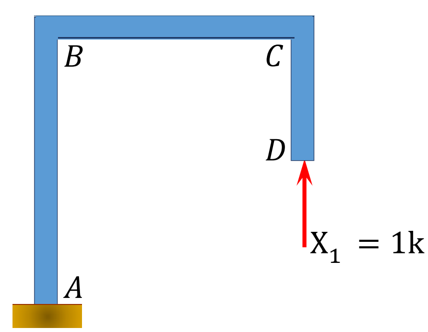
Example 10.7

Using the method of consistent deformation, determine the support reactions of the truss shown in Figure 10.9a. EI = constant.

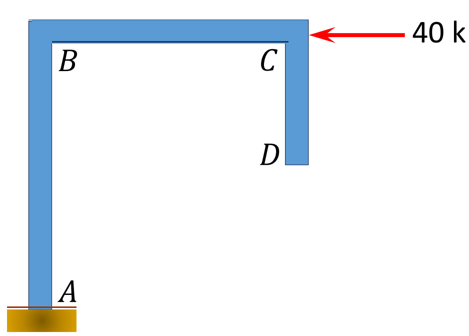


(a) Actual frame

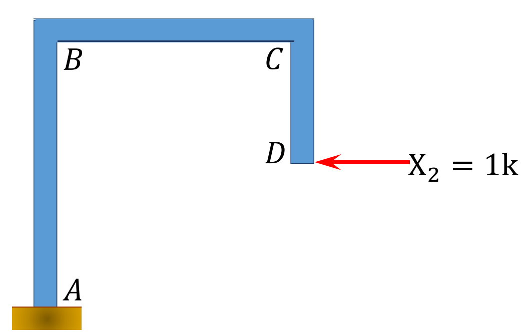
Fig. 10.9. Truss.



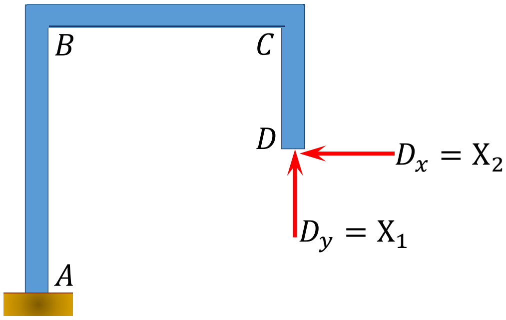
(d) Primary frame subjected to $D_y = 1$.



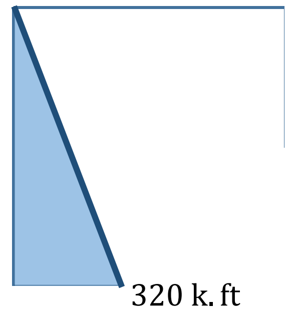
(b) Primary frame subjected to external load



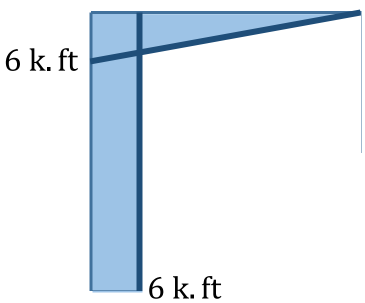
(e) Primary frame subjected to $D_x = 1$.



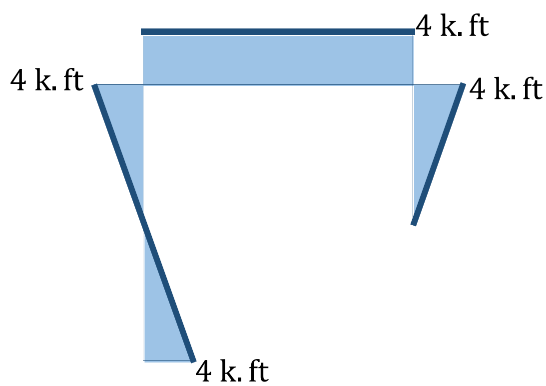
(c) Redundants D_y and D_x
Applied on primary frame.



(f) Bending moment for primary
frame sujected to external loading



(g) Bending moment for primary
frame due to $D_y = 1$ k



(h) Bending moment for primary
frame due to $D_x = 1$ k

Solution

Classification of structure. There are five unknown reactions in the beam. Thus, the degree of indeterminacy of the structure is two.

Choice of primary structure. The two reactions of the pin support at *D* are chosen as the redundant reactions, therefore the primary structure is a cantilever beam subjected to a horizontal load at *C*, as shown in Figure 10.9b. The primary structure loaded with the redundant unknowns is shown in Figure 10.9d and Figure 10.9e.

Compatibility equation. The number of compatibility equations is two, since there are two redundant unknowns. The equations are written as follows:

$$\Delta_{1P} + X_1\delta_{11} + X_2\delta_{12} = 0$$

$$\Delta_{2P} + X_1\delta_{21} + X_2\delta_{22} = 0$$

The first number of the subscript in the flexibility coefficients indicates the direction of the deflection, while the second number or letter indicates the force causing the deflection. The coefficients are computed using the graph multiplication method, as follows:

$$\Delta_{1P} = \frac{1}{EI}\left(\frac{1}{2} \times 8 \times 320\right)(6) = \frac{7680}{EI}$$

$$\Delta_{2P} = \frac{1}{EI}\left(-\frac{1}{2} \times 4 \times 4\right)(53.33) + \left(\frac{1}{2} \times 4 \times 4\right)(266.8) = \frac{1707.76}{EI}$$

$$\delta_{11} = \frac{1}{EI}\left(\frac{1}{2} \times 6 \times 6\right)(4) + (6 \times 8)(6) = \frac{360}{EI}$$

$$\delta_{12} = \delta_{21} = \frac{1}{EI}\left(-\frac{1}{2} \times 4 \times 4\right)(6) + \left(\frac{1}{2} \times 4 \times 4\right)(6) - \left(\frac{1}{2} \times 6 \times 6\right)(4) = -\frac{72}{EI}$$

$$\delta_{22} = \frac{1}{EI}(3)\left(\frac{1}{2} \times 4 \times 4\right)(2.67) + (4 \times 6)(4) = \frac{160.08}{EI}$$

Substituting the flexibility coefficients into the compatibility equation suggests the following two equations with two unknowns:

$$7680 + 360X_1 - 72X_2 = 0$$

$$1707.76 - 72X_1 + 160.08X_2 = 0$$

Solving both equations simultaneously suggests the following:

$X_1 = D_y = 25.79\text{k}$

$X_2 = D_x = 22.27\text{k}$

Determination of the reactions at support *A*.

$$\sum M_A = 0{:}\ (25.79)(6) + (22.27)(16) + (21.46)(4) + M_A = 0$$

$$M_A = 25.6\ \text{k.ft}$$

$$\sum F_y = 0{:}\ A_y - 30 + 21.46 - 4.83 = 0$$

$$A_y = 13.37\ \text{k}$$

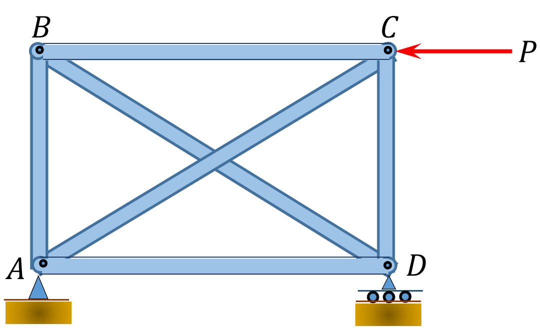
10.4 Analysis of Indeterminate Trusses

The procedure for the analysis of indeterminate trusses is similar to that followed in the analysis of beams. For trusses with external redundant restraints, the procedure entails determining the degree of indeterminacy of the structure, selecting the redundant reactions, writing the compatibility equations, determining the deflection due to the applied load and the one due to a unit redundant reaction force applied to the primary structure, and solving the compatibility equation(s) to determine the redundant reactions. For trusses with internal redundant members, the procedure involves selecting the redundant members, cutting the redundant members and depicting each of them as a pair of forces in the primary structure, and then applying the condition of compatibility to determine the axial forces in the redundant members. Consider the truss below for an example. This truss is indeterminate to the first degree. Members *AC* and *BD* of the truss are two separate overlapping members. Either of these members can be considered redundant, since the primary structure obtained after the removal of either of them will remain stable. Selecting *BD* as the redundant member, cutting through it and applying a pair of forces on the cut surface, and then indicating that the displacement of the truss at the cut surface is zero suggests the following compatibility expression:

$$\Delta_{BD} + F_{BD}\delta_{BD} = 0 \qquad (10.7)$$

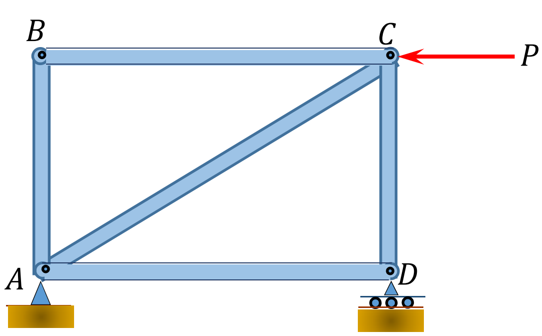
where

Δ_{BD}= the relative displacement of the cut surface due to the applied load.
δ_{BD} = the relative displacement of the cut surface due to an applied unit redundant load on the cut surface.

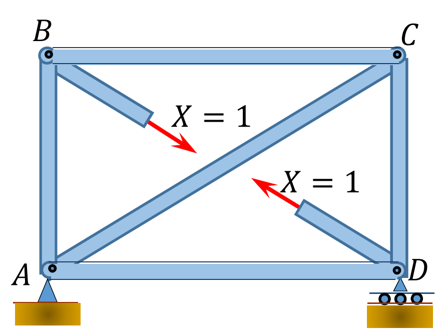


(a) Actual Structure

Fig. 10.10



(b) Primary Structure



(c) Redundant $X = 1$ applied

The flexibility coefficients for the compatibility equation for the indeterminate truss analysis is computed as follows:

$$\Delta_{XP} = \sum \frac{FfL}{AE}$$

$$\delta_{XX} = \sum \frac{f^2 L}{AE} \qquad (10.8)$$

where

Δ_{XP}= the displacement at a joint X or member of the primary truss due to applied external load.

δ_{X1} = the displacement at joint X or member of the primary truss due to the unit redundant force.

F = axial force in the truss members due to the applied external load that causes the displacement Δ.

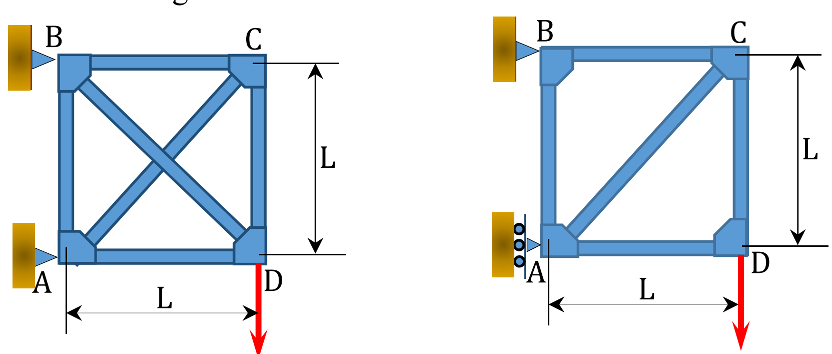
f = axial forces in truss members due to the applied unit redundant load that causes the displacement δ.

L = length of member.

A = cross sectional area of a member.

Example 10.8

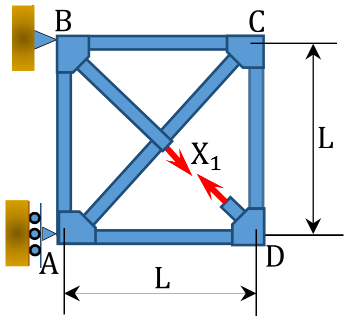
Using the method of consistent deformation, determine the axial force in all the members of the truss shown in Figure 10.11a. EA = constant. .



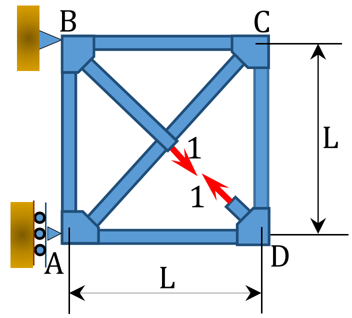
(a) Actual truss

(b) Primary truss subjected to external load

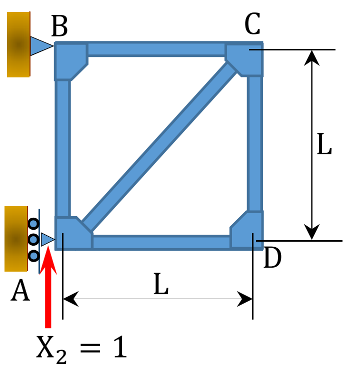
Fig. 10.11. Truss.



(c)Redundant X_1 applied on primary truss



(d) Primary truss subjected to redundant $X_1 = 1$



(e) Primary truss subjected redundant $X_2 = 1$

Solution

Determining support reactions in the primary structure.

$$+\circlearrowleft \sum M_B = 0$$
$$-PL + A_x L = 0$$
$$A_x = P$$

$$+\rightarrow \sum F_x = 0$$
$$-P + B_x = 0$$
$$B_x = P$$

$$+\uparrow \sum F_y = 0$$
$$-P - B_y = 0$$
$$B_y = P$$

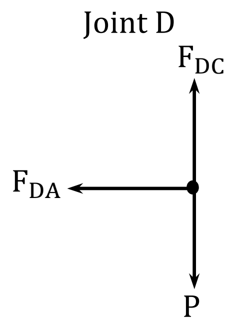
Compatibility Equation.

$$\Delta_{1P} + X_1\delta_{11} + X_2\delta_{12} = 0$$

$$\Delta_{2P} + X_1\delta_{21} + X_2\delta_{22} = 0$$

Determining forces in members due to applied external load.

Joint *D*.

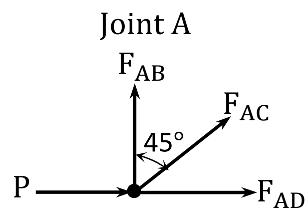


$$+\uparrow \sum F_y = 0: F_{DC} - P = 0$$

$$F_{DC} = P$$

$$+\rightarrow \sum F_x = 0: F_{DA} = 0$$

Joint *A*.



$$\rightarrow \sum F_x = 0: F_{AC}\cos 45° + F_{AD} + P = 0$$

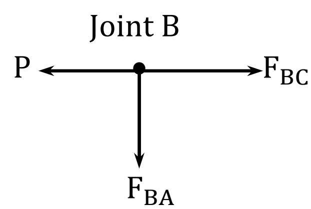
$$F_{AC} = -\frac{P}{\cos 45°} = -1.414P$$

$$+\uparrow \sum F_y = 0: F_{AB} + F_{AC}\cos 45° = 0$$

$$F_{AB} = -(-1.414P)\cos 45° = P$$

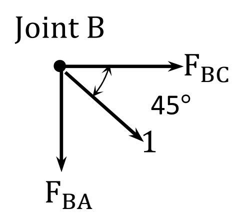
Joint *B*.

$$+\rightarrow \sum F_x = 0: -P + F_{BC} = 0;\ F_{BC} = P$$



Determining forces in members due to redundant $F_{BD} = 1$.

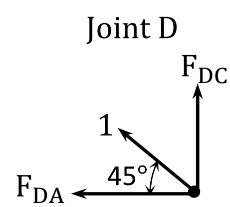
Joint *B*.



$$+\rightarrow \sum F_x = 0: 1\cos 45° + F_{BC} = 0;\ F_{BC} = -\cos 45° = -0.7071$$

$$+\uparrow \sum F_y = 0: -1\cos 45° - F_{BA} = 0;\ F_{BA} = -0.7071$$

Joint *D*.

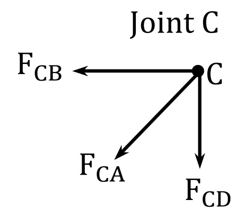


$$+\rightarrow \sum F_x = 0: -1\cos 45° - F_{DA} = 0;\ F_{DA} = -\cos 45° = -0.7071$$

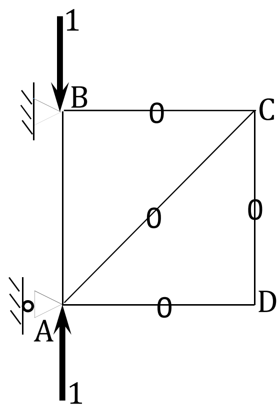
$$+\uparrow \sum F_y = 0{:}\ 1\cos 45° + F_{DC} = 0;\ F_{DC} = -0.7071$$

Joint *C*.

$$+\uparrow \sum F_y = 0{:}\ -F_{CA}\cos 45° - F_{CD} = 0;\ F_{CA} = 1$$



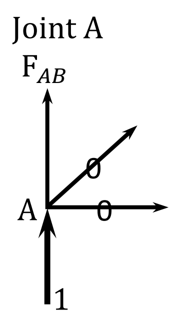
Determining forces in members due to redundant $A_y = 1$.



Joint *A*.

$$+\uparrow \sum F_y = 0{:}\ 1 + F_{AB} = 0$$

$$F_{AB} = -1$$



The determination of the member-axial forces can be conveniently performed in a tabular form, as shown in Table 10.3.

Table 10.3.

Member	Length	N	n_{BD}	n_A	$n_{BD}^2 L$	$n_A^2 L$	$n_{BD} n_A L$	$N n_{BD} L$	$N n_A L$
AB	L	P	−0.7071	−1	0.5L	L	0.7071L	−0.7071PL	−PL
AC	1.414L	−1.414P	1	0	1.414L	0	0	−2PL	0
AD	L	0	−0.7071	0	0.5L	0	0	0	0
BC	L	P	−0.7071	0	0.5L	0	0	−0.7071PL	0
BD	1.414L	0	1	0	1.414L	0	0	0	0
CD	L	P	−0.7071	0	0.5L	0	0	−0.7071PL	0
Total					4.828*L*	*L*	0.7071L	−4.12PL	−PL

$\Delta_{1P} = -\frac{4.12PL}{EA}$

$\Delta_{2P} = -\frac{PL}{EA}$

$\delta_{11} = \frac{4.828L}{EA}$

$\delta_{12} = \delta_{12} = \frac{0.7071L}{EA}$

$\delta_{22} = \frac{L}{EA}$

Substituting the flexibility coefficient into the compatibility equations and solving the simultaneous equations suggests the following:

$$-4.12P + 4.828X_1 + 0.7071X_2 = 0$$

$$-P + 0.7071X_1 + X_2 = 0$$

$X_1 = F_{BD} = 0.79\text{P}$

$X_2 = A_y = 0.44\text{P}$

The axial forces in members are as follows:

$F_{AB} = P + (0.79P)(-0.7071) + (0.44P)(-1) = 0.0014P$

$F_{AC} = -1.414P + (1)(0.79P) = -0.624P$

$F_{AD} = (-0.7071)(0.79P) = -0.559P$

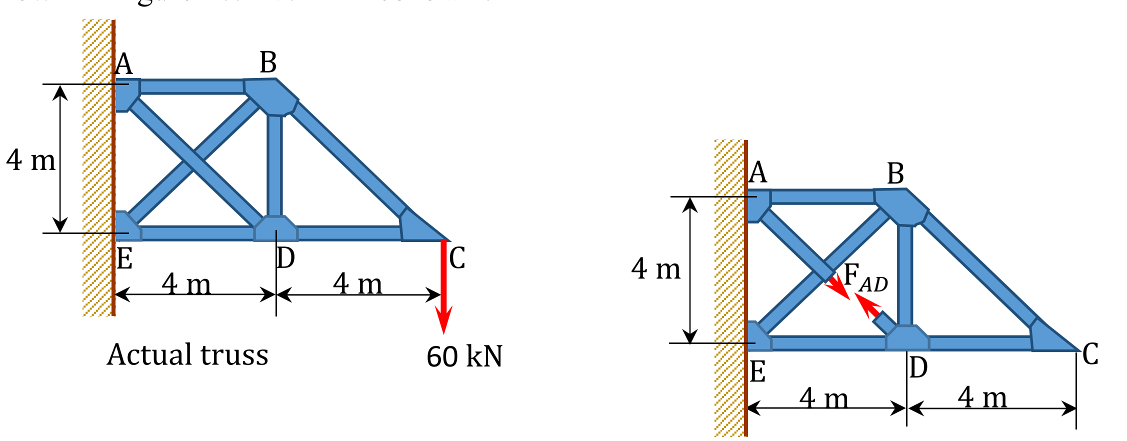
$F_{BC} = P + (-0.7071)(0.79P) = 0.441P$

$F_{BD} = 0.79P$

$F_{CD} = P + (-0.7071)(0.79P) = 0.441P$

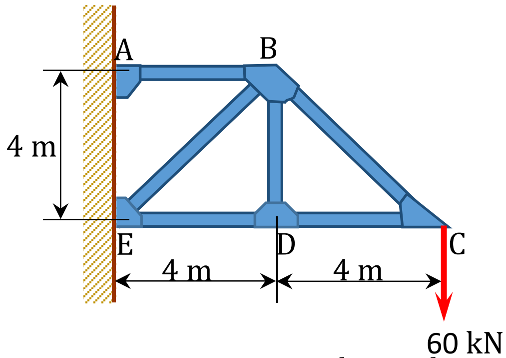
Example 10.9

Using the method of consistent deformation, determine the axial force in member *AD* of the truss shown in Figure 10.12a. *EA* = constant.

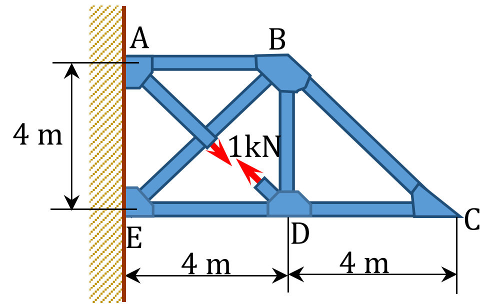


Redundant $F_{AD} = X_1$ applied on primary truss

Fig. 10.12. Truss.



Primary truss subjected to external load

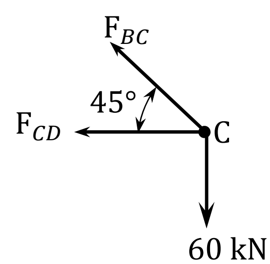


Primary truss subjected to $F_{AD} = X_1 = 1$

Solution

Determination of axial forces in members due to applied external loads.

Joint C



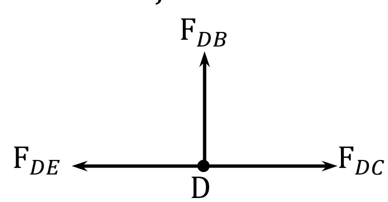
$$+\uparrow \sum F_y = 0: F_{BC}\sin 45° - 60 = 0$$

$$F_{BC} = 84.85 \text{ kN}$$

$$+\rightarrow \sum F_x = 0: -F_{CD} - F_{BC}\cos 45° = 0$$

$$F_{CD} = -84.85\cos 45° = -60\text{kN}$$

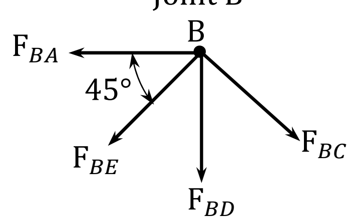
Joint D



$$+\uparrow \sum F_y = 0: F_{DB} = 0$$

$$\rightarrow \sum F_x = 0: F_{DE} = F_{DC} = -60 \text{ kN}$$

Joint B



$$+\uparrow \sum F_y = 0: -F_{BE}\cos 45° - F_{BC}\cos 45° = 0$$

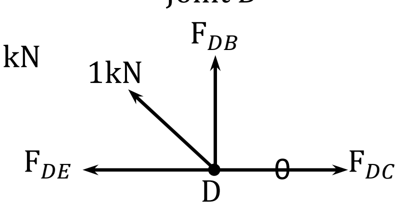
$$F_{BE} = -F_{BC} = -84.85 \text{ kN}$$

$$+\rightarrow \sum F_x = 0: -F_{BA} - F_{BE}\cos 45° + F_{BC}\cos 45° = 0$$

$$F_{BA} = 120 \text{ kN}$$

Determining forces in members due to redundant $F_{AD} = 1$.

Joint D



$$+\uparrow \sum F_y = 0: \cos 45° + F_{DB} = 0$$

$$F_{DB} = -\cos 45° \text{ kN} = -0.7071 \text{ kN}$$

$$+\rightarrow \sum F_x = 0: -F_{DE} - \cos 45° = 0$$

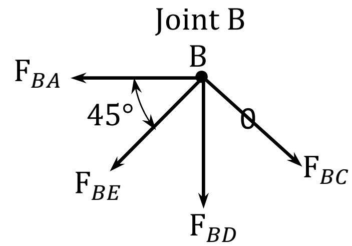
$F_{DE} = -0.7071 \text{ kN}$

$$+\uparrow \sum F_y = 0: -F_{BE} \cos 45° - F_{BD} = 0$$

$$F_{BE} = -\frac{F_{BD}}{\cos 45°} = \frac{0.7071}{0.7071} = 1 \text{ kN}$$

$$+\rightarrow \sum F_x = 0: -F_{BA} - F_{BE} \cos 45° = 0$$

$F_{BA} = -0.7071 \text{ kN}$



The determination of the member-axial forces can be conveniently performed in a tabular form, as shown in Table 10.4.

Table 10.4.

Member	Length (m)	N (kN)	n_{AD} (kN)	$Nn_{AD}L$	$n_{AD}^2 L$
AB	4	120	−0.7071	−339.41	2.0
AD	5.66	0	1	0	5.66
BE	5.66	−84.85	1	−480.25	5.66
BD	4	0	−0.7071	0	2
BC	5.66	84.85	0	0	0
CD	4	−60	0	0	0
DE	4	−60	−0.7071	169.7	2
				−649.96	17.32

Compatibility equation.

$$\Delta_{1P} + X_1 \delta_{11} = 0$$

$$F_{AD} = X_1 = -\frac{\Delta_{1p}}{\delta_{11}} = \frac{649.96}{17.32} = 37.53 \text{ kN}$$

Chapter Summary

Force method: The force method or the method of consistent deformation is based on the equilibrium of forces and compatibility of structures. The method entails first selecting the unknown redundants for the structure and then removing the redundant reactions or members to obtain the primary structure.

Compatibility equations: The compatibility equations are formulated and used together with the equations of equilibrium to determine the unknown redundants. The number of the compatibility equations must match the number of the unknown redundants. Once the unknown redundants are determined, the structure becomes determinate. Methods of computation of compatibility or flexibility coefficients, such as the method of integration, the graph multiplication method, and the use of deflection tables, are solved in the chapter.

Mohr integral for computation of flexibility coefficients for beams and frames:

$$\Delta_{BP} = \int \frac{Mm}{EI} dx$$

$$\delta_{BB} = \int \frac{m^2}{EI} dx$$

$$\theta_{AP} = \int \frac{Mm_\theta}{EI} dx$$

$$\alpha_{AA} = \int \frac{m_\theta^2}{EI} dx$$

Maxwell-Betti law of reciprocal deflections: The Maxwell-Betti law helps reduce the computational efforts required to obtain the flexibility coefficients for the compatibility equations. This law states that the linear displacement at point *A* due to a unit load applied at *B* is equal in magnitude to the linear displacement at point *B* due to a unit load applied at *A* for a stable elastic structure. This law is expressed as follows:

$$\delta_{AB} = \delta_{BA}$$

Practice Problems

10.1 Using the method of consistent deformation, compute the support reactions and draw the shear force and the bending moment diagrams for the beams shown in Figures P10.1 through P10.4. Choose the reaction at the interior support *B* as the unknown redundant.

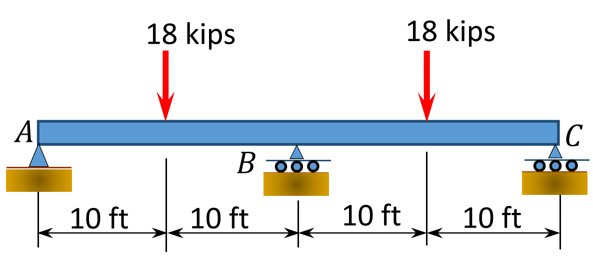


Fig. P10.1. Beam. EI = constant

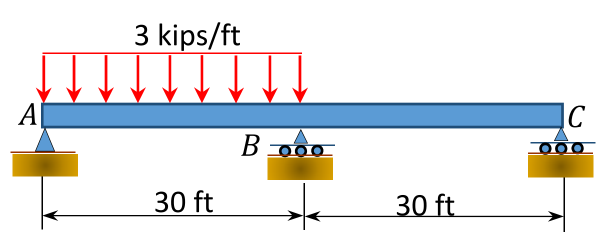


Fig. P10.2. Beam. EI = constant

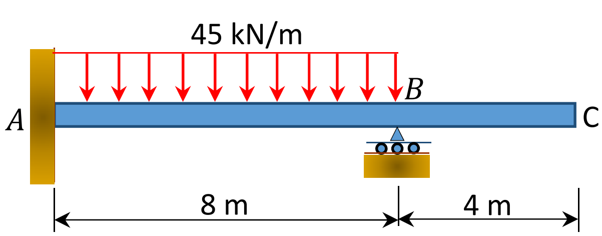


Fig. P10.3. Beam. EI = constant

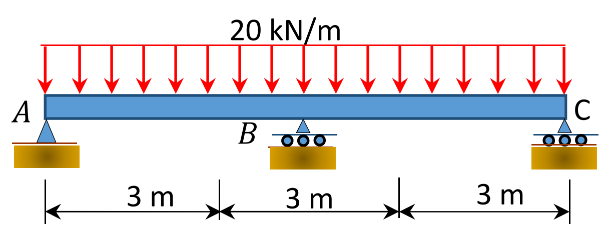


Fig. P10.4. Beam. EI = constant

10.2 Using the method of consistent deformation, compute the support reactions and draw the shear force and the bending moment diagrams for the frames shown in Figures P10.5 through P10.8. Choose the reaction(s) at any of the supports as the unknown redundant(s). EI = constant.

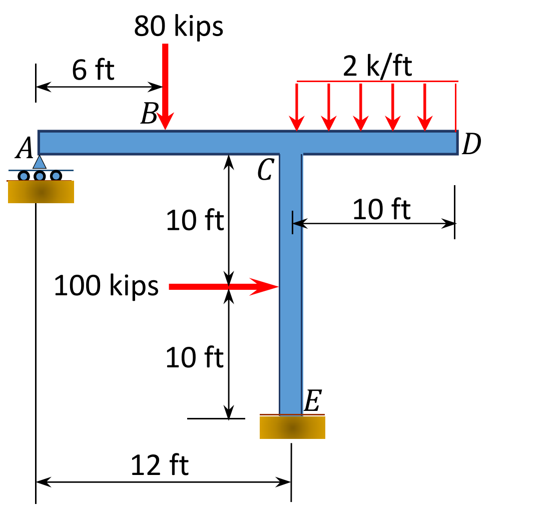


Fig. P10.5. Frame.

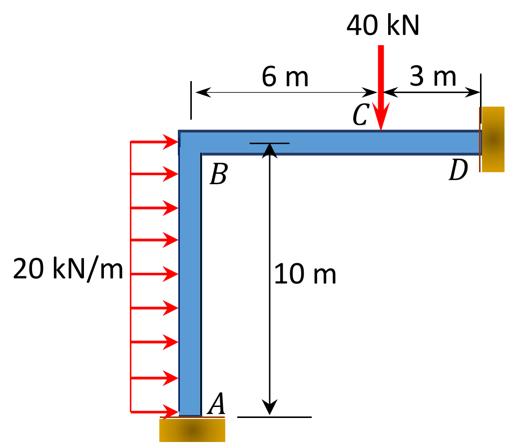


Fig. P10.6. Frame.

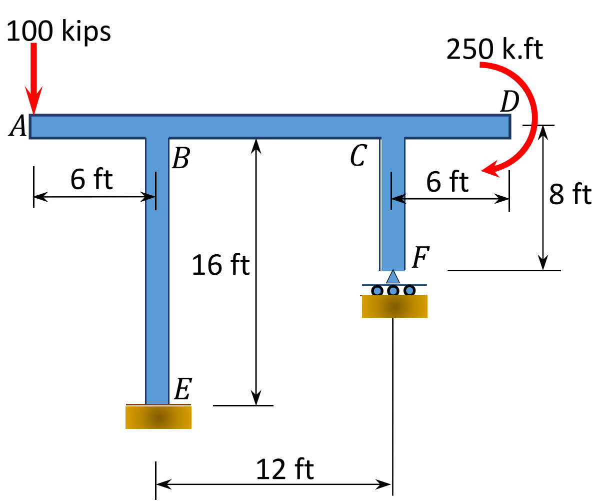


Fig. P10.7. Frame.

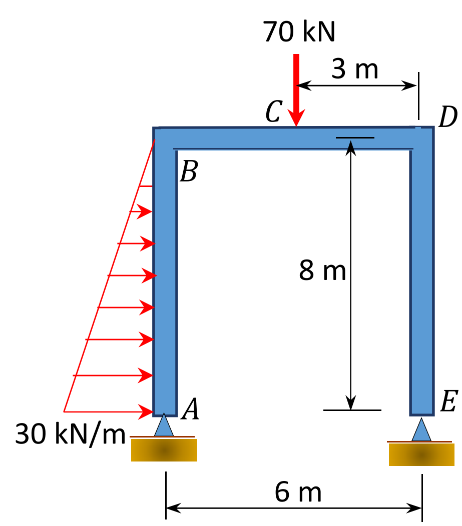


Fig. P10.8. Frame.

10.3 Using the method of consistent deformations, determine the reactions and the axial forces in the members of the trusses shown in Figures P10.9 through P10.13.

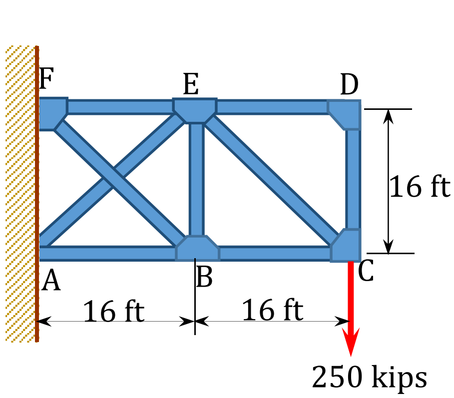


Fig. P10.9. Truss. EA = constant

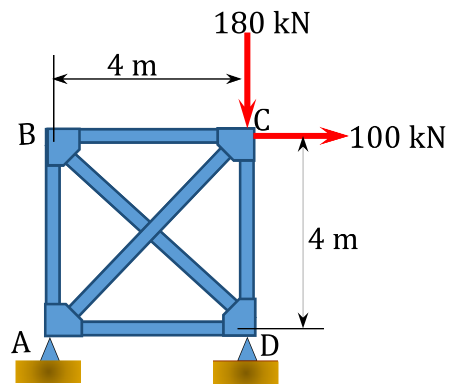


Fig. P10.10. Truss. EA = constant

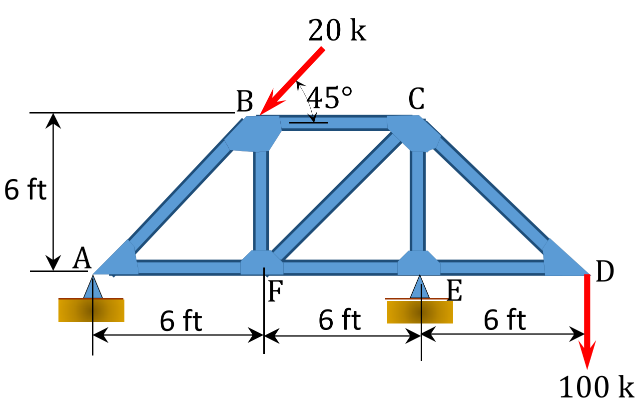


Fig. P10.11. Truss. EA = constant

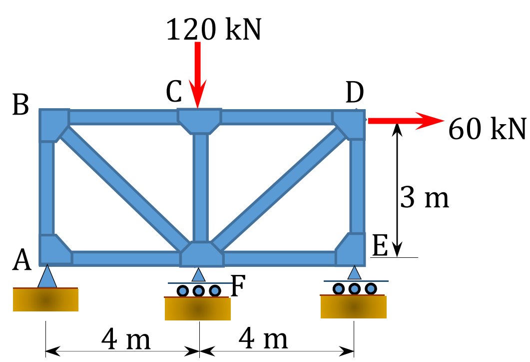


Fig. P10.12. Truss. EA = constant

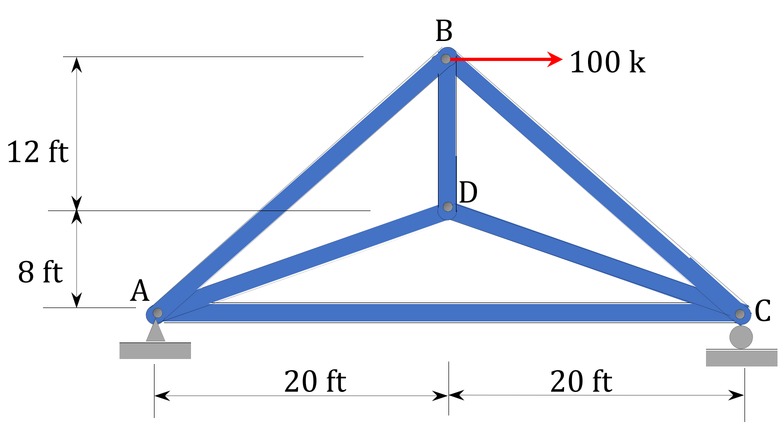


Fig. 10.13. Truss. EA = constant

Chapter 11

Slope-Deflection Method of Analysis of Indeterminate Structures

11.1 Introduction

In 1915, George A. Maney introduced the slope-deflection method as one of the classical methods of analysis of indeterminate beams and frames. The method accounts for flexural deformations, but ignores axial and shear deformations. Thus, the unknowns in the slope-deflection method of analysis are the rotations and the relative joint displacements. For the determination of the end moments of members at the joint, this method requires the solution of simultaneous equations consisting of rotations, joint displacements, stiffness, and lengths of members.

11.2 Sign Conventions

An end moment M is considered positive if it tends to rotate the member clockwise and negative if it tends to rotate the member counter-clockwise. The rotation θ of a joint is positive if its tangent turns in a clockwise direction. The rotation of the chord connecting the ends of a member $\left(\frac{\Delta}{l}\right)$, the displacement of one end of a member relative to the other, is positive if the member turns in a clockwise direction.

11.3 Derivation of Slope-Deflection Equations

To derive the slope-deflection equations, consider a beam of length L and of constant flexural rigidity EI loaded as shown in Figure 11.1a. The member experiences the end moments M_{AB} and M_{BA} at A and B, respectively, and undergoes the deformed shape shown in Figure 11.1b, with the assumption that the right end B of the member settles by an amount Δ. The end moments are the summation of the moments caused by the rotations of the joints at the ends A and B (θ_A and θ_B) of the beam, the chord rotation $\left(\psi = \frac{\Delta}{l}\right)$, and the fixity at both ends referred to as fixed end moments (M_{AB}^F and M_{BA}^F).

The rotations at the joints of the beam can be expressed mathematically as follows:

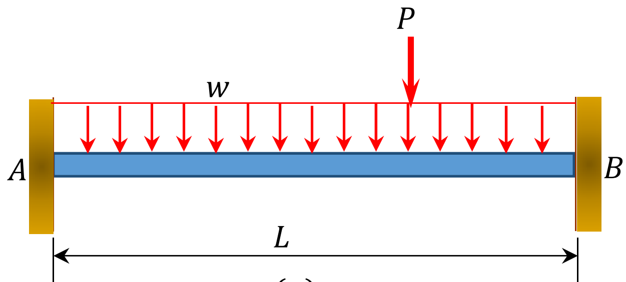
$$\theta_A = \beta_A + \psi \tag{11.1}$$

$$\theta_B = \beta_B + \psi \tag{11.2}$$

where

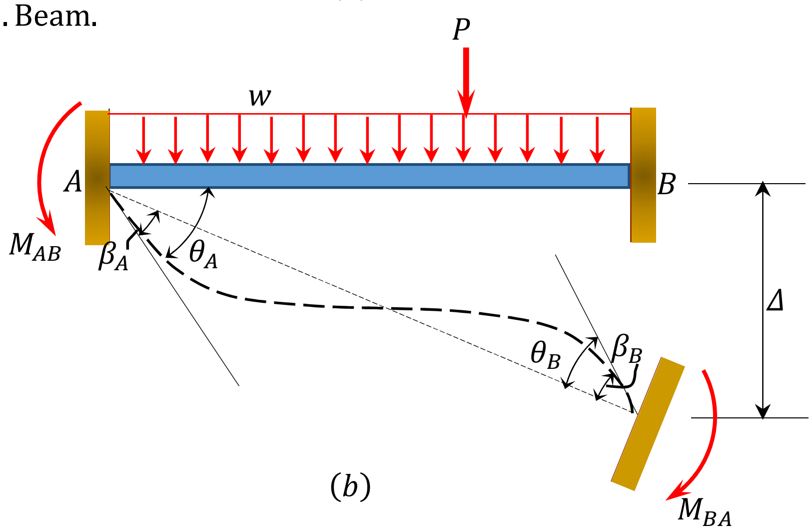
β_A, β_B = end rotations caused by moments M_{AB} and M_{BA}, respectively.

$\psi =$ chord rotation caused by settlement of end B.

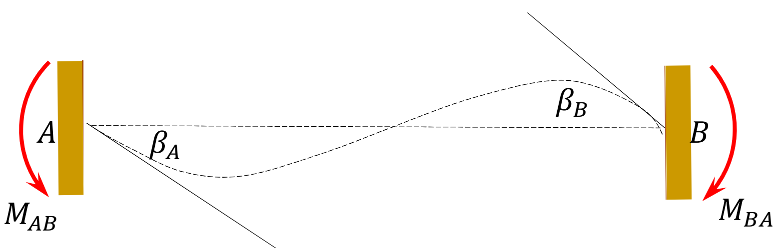


(a)

Fig. 11.1. Beam.

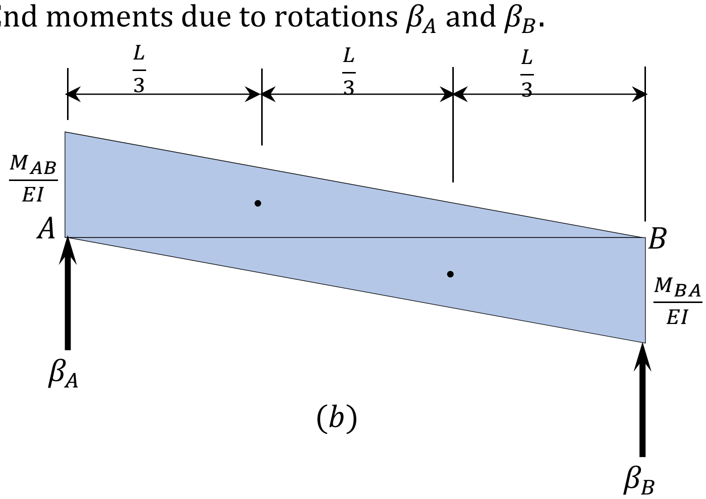


(b)



(a)

Fig. 11.2. End moments due to rotations β_A and β_B.



(b)

According to the moment-area theorem, the change in slope for a particular beam equals the end shear force of the beam when it is loaded with the $\frac{M}{EI}$ diagram. Thus, for the beam under consideration, the rotations β_A and β_B, shown in Figure 11.2, are obtained as follows:

$$\curvearrowright +\sum M_B = 0; -\beta_A L + \left(\frac{1}{2}\right)\left(\frac{M_{AB}}{EI}\right)(L)\left(\frac{2}{3}L\right) - \left(\frac{1}{2}\right)\left(\frac{M_{BA}}{EI}\right)(L)\left(\frac{1}{3}L\right) = 0$$

$$\beta_A = \frac{\left(\frac{1}{2}\right)\left(\frac{M_{AB}}{EI}\right)(L)\left(\frac{2}{3}L\right) - \left(\frac{1}{2}\right)\left(\frac{M_{BA}}{EI}\right)(L)\left(\frac{1}{3}L\right)}{L}$$

$$= \frac{L}{6EI}(2M_{AB} - M_{BA}) \quad (11.3)$$

Similarly, taking the moment about end *A* to determine β_B suggests the following:

$$\curvearrowright +\sum M_A = 0; \beta_B L + \left(\frac{1}{2}\right)\left(\frac{M_{BA}}{EI}\right)(L)\left(\frac{2}{3}L\right) - \left(\frac{1}{2}\right)\left(\frac{M_{AB}}{EI}\right)(L)\left(\frac{1}{3}L\right) = 0$$

$$\beta_B = \frac{\left(\frac{1}{2}\right)\left(\frac{M_{BA}}{EI}\right)(L)\left(\frac{2}{3}L\right) - \left(\frac{1}{2}\right)\left(\frac{M_{AB}}{EI}\right)(L)\left(\frac{1}{3}L\right)}{L}$$

$$= \frac{L}{6EI}(2M_{BA} - M_{AB}) \quad (11.4)$$

Solving equations 11.3 and 11.4 suggests the following:

$$M_{AB} = \frac{4EI}{L}\beta_A + \frac{2EI}{L}\beta_B \quad (11.5)$$

$$M_{BA} = \frac{2EI}{L}\beta_A + \frac{4EI}{L}\beta_B \quad (11.6)$$

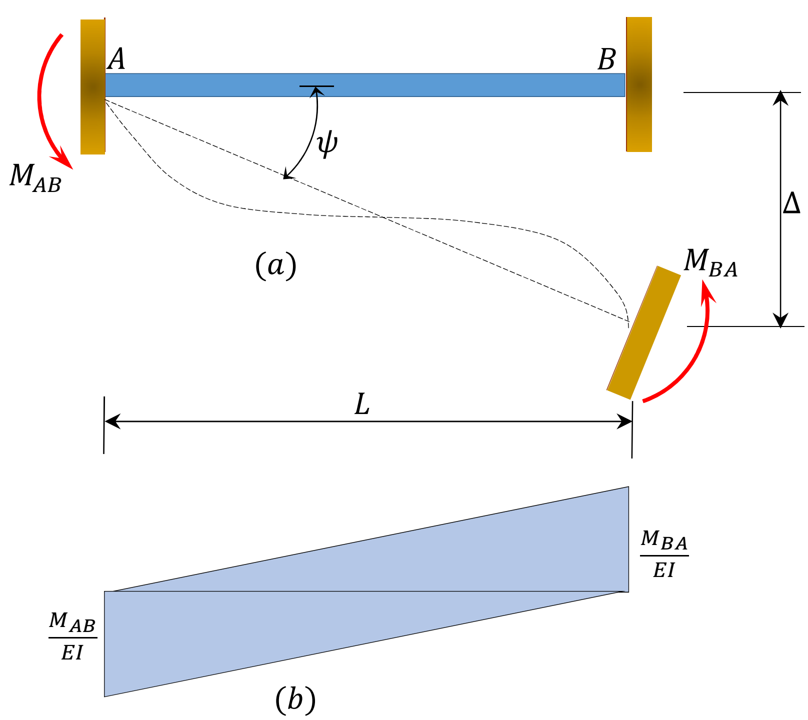


Fig. 11.3. End moments due to end rotations (β_A and β_B) and chord rotation (ψ).

Solving equations 11.1 and 11.2 for β_A and β_B and substituting them into equations 11.5 and 11.6 suggests the following:

$$M_{AB} = \frac{4EI}{L}(\theta_A - \psi) + \frac{2EI}{L}(\theta_B - \psi) \tag{11.7}$$

$$M_{BA} = \frac{2EI}{L}(\theta_A - \psi) + \frac{4EI}{L}(\theta_B - \psi) \tag{11.8}$$

Putting $\psi = \frac{\Delta}{L}$ into equations 10.10 and 10.11 suggests the following:

$$M_{AB} = \frac{4EI}{L}\theta_A + \frac{2EI}{L}\theta_B - \frac{6EI}{L^2}\Delta \tag{11.9}$$

$$M_{BA} = \frac{2EI}{L}\theta_A + \frac{4EI}{L}\theta_B - \frac{6EI}{L^2}\Delta \tag{11.10}$$

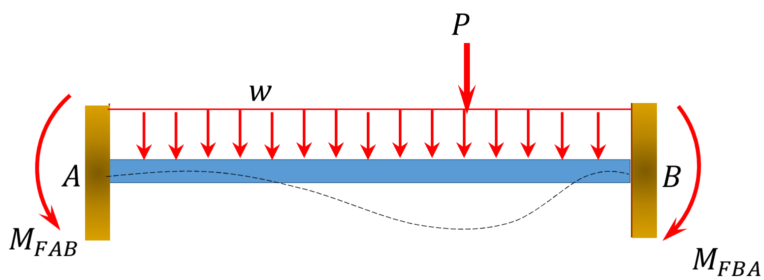


Fig. 11.4. End moment due to end rotations (β_A and β_B) , chord rotation (ψ), and fixed-end moments (M_{AB}^F and M_{BA}^F).

The final end moments can then be computed as the summation of the moments caused by slopes, deflections, and fixed-end moments, as follows:

$$\begin{aligned} M_{AB} &= 2EK(2\theta_A + \theta_B - 3\psi) + M_{AB}^F \\ M_{BA} &= 2EK(\theta_A + 2\theta_B - 3\psi) + M_{BA}^F \end{aligned} \tag{11.11}$$

where

$K = \frac{I}{L}$ = stiffness factor.

11.4 Modification for Pin-Supported End Span

The analysis of beams or frames supported by a pin or roller at the far end of the span is simplified by using the modified slope-deflection equation derived below. Using the modified equation

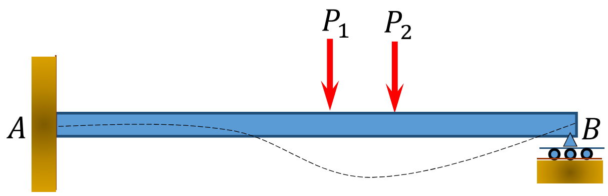


Fig. 11.5. Propped cantilever beam.

reduces the amount of computational work, as the equation is applied only once to the span with a pin or roller at the far end.

Consider the propped cantilever beam shown in Figure 11.5. The slope-deflection equations for the end moments are as follows:

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3\psi) + M^F_{AB} \quad (11.12)$$

$$M_{BA} = 0 = 2EK(\theta_A + 2\theta_B - 3\psi) + M^F_{BA} \quad (11.13)$$

Solving equation 11.13 for θ_B and substituting it into equation 11.12 suggests the following:

$$M_{AB} = 3EK(\theta_A - \psi) + \left(M^F_{AB} - \frac{M^F_{BA}}{2}\right) \quad (11.14)$$

Equation 11.14 is the modified slope-deflection equation when the far end is supported by a pin or roller.

11.5 Analysis of Indeterminate Beams

The procedure for the analysis of indeterminate beams by the slope-deflection method is summarized below.

Procedure for Analysis of Indeterminate Beams and Non-Sway Frames by the Slope-Deflection Method

- Determine the fixed-end moments for the members of the beam.
- Determine the rotations of the chord if there is any support settlement.
- Write the slope-deflection equation for the members' end moments in terms of unknown rotations.
- Write the equilibrium equations at each joint that is free to rotate in terms of the end moments of members connected at that joint.
- Solve the system of equations obtained simultaneously to determine the unknown joint rotations.
- Substitute the computed joint rotations into the equations obtained in step 3 to determine the members' end moments.
- Draw a free-body diagram of the indeterminate beams indicating the end moments at the joint.
- Draw the shearing force diagrams of the beam by considering the free-body diagram of each span of the beam in the case of a multi-span structure.

11.6 Analysis of Indeterminate Frames

Indeterminate frames are categorized as frames with or without side-sway. A frame with side-sway is one that permits a lateral moment or a swaying to one side due to the asymmetrical nature of its structure or loading. The analysis of frames without side-sway is similar to the analysis of beams considered in the preceding section, while the analysis of frames with side-sway requires taking into consideration the effect of the lateral movement of the structure.

11.6.1 Analysis of Frames with Side-Sway

Consider the frame shown in Figure 11.6 for an illustration of the effect of side-sway on a frame. Due to the asymmetrical application of the loads, there will be a lateral displacement Δ to the right at B and C, which subsequently will cause chord rotations $\psi_{AB}\left(\frac{\Delta}{L_{AB}}\right)$ and $\psi_{DC}\left(\frac{\Delta}{L_{DC}}\right)$ in columns AB and DC, respectively. These rotations must be considered when writing the slope-deflection equations for the columns, as will be demonstrated in the solved examples.

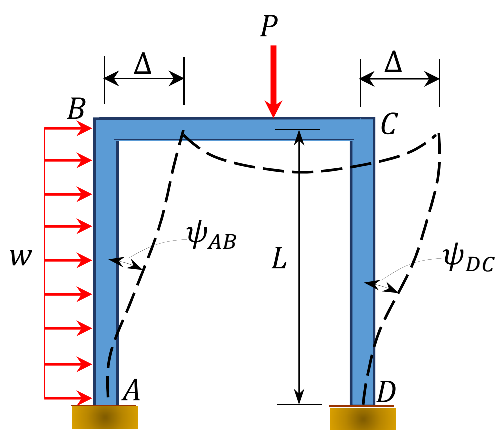


Fig. 11.6. Frame.

Example 11.1

Using the slope-deflection method, determine the end moments and the reactions at the supports of the beam shown in Figure 11.7a and draw the shearing force and the bending moment diagrams. EI = constant.

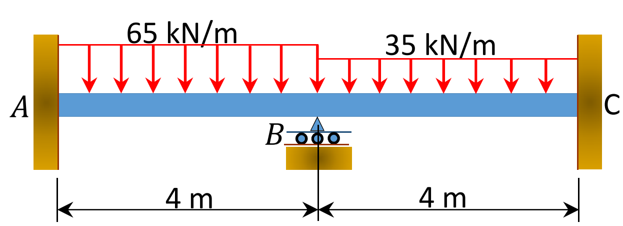


Fig. 11.7. Beam. (a)

Solution

Fixed-end moments.

The Fixed-end moments (FEM) using Table 11.1 are computed as follows:

$$FEM_{AB} = -\frac{wL^2}{12} = -\frac{65\times4^2}{12} = -86.67 \text{ kN.m}$$

$$FEM_{BA} = \frac{wL^2}{12} = 86.67 \text{ kN.m}$$

$$FEM_{BC} = -\frac{35\times4^2}{12} = -46.67 \text{ kN.m}$$

$$FEM_{CB} = 46.67 \text{ kN.m}$$

Slope-deflection equations. As $\theta_A = \theta_C = 0$ due to fixity at both ends and $\psi_{AB} = \psi_{BC} = 0$ as no settlement occurs, equations for member end moments are expressed as follows:

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi) + FEM_{AB}$$

$$2EK\theta_B - 86.67 \qquad (1)$$

$$M_{BA} = \frac{2EI}{L}(\theta_A + 2\theta_B - 3\psi) + FEM_{BA}$$

$$4EK\theta_B + 86.67 \qquad (2)$$

$$M_{BC} = \frac{2EI}{L}(2\theta_B + \theta_C - 3\psi) + FEM_{BC}$$

$$4EK\theta_B - 46.67 \qquad (3)$$

$$M_{CB} = \frac{2EI}{L}(\theta_B + 2\theta_C - 3\psi) + FEM_{CB}$$

$$2EK\theta_B + 46.67 \qquad (4)$$

Joint equilibrium equation.

Equilibrium equation at joint B is as follows:

$$\sum M_B = M_{BA} + M_{BC} = 0$$

$$4EK\theta_B + 86.67 + 4EK\theta_B - 46.67 = 0$$

$$\theta_B = -\frac{5}{EK}$$

Final end moments.

Substituting $\theta_B = -\frac{5}{EK}$ into equations 1, 2, 3, and 4 suggests the following:

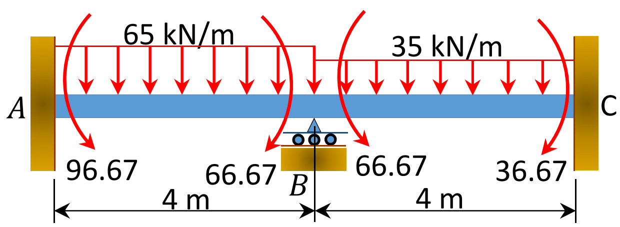
$$M_{AB} = 2EK\left(-\frac{5}{EK}\right) - 86.67 = -96.67 \text{ kN.m}$$

$$M_{BA} = 4EK\left(-\frac{5}{EK}\right) + 86.67 = 66.67 \text{ kN.m}$$

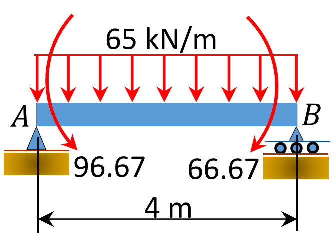
$$M_{BC} = 4EK\left(-\frac{5}{EK}\right) - 46.67 = -66.67 \text{ kN.m}$$

$$M_{CB} = 2EK\left(-\frac{5}{EK}\right) + 46.67 = 36.67 \text{ kN.m}$$

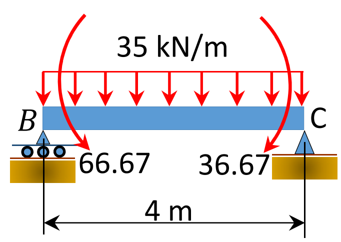
Shearing force and bending moment diagrams.



(b)



(c)



(d)

Shear force and bending moment for segment AB.

First compute the reaction at support A, as follows:

$\curvearrowleft + \sum M_B = 0: -4A_y + 96.67 + (65)(4)(2) - 66.67 = 0$

$A_y = 137.5 \text{ kN}$

Calculate the shear force, as follows:

$V = 137.5 - 65x$

When $x = 0, V = 137.5 \text{ kN}$

When $x = 4$ m, $V = -122.5$ kN

Find the moment, as follows:

$$M = 137.5x - \frac{(65)(x)^2}{2} - 96.67$$

When $x = 0$, $M = -96.67$ kN.m

When $x = 4$ m, $M = -66.67$ kN.m

Shear force and bending moment for segment *AB*.

First determine the reaction at *B*, as follows:

$$\curvearrowleft + \sum M_C = 0: -4B_y + 66.67 + (35)(4)(2) - 36.67 = 0$$

$$B_y = 77.5 \text{ kN}$$

Calculate the shear force, as follows:

$$V = 77.5 - 35x$$

When $x = 0$, $V = 77.5$ kN.

When $x = 4$ m, $V = -62.5$ kN.

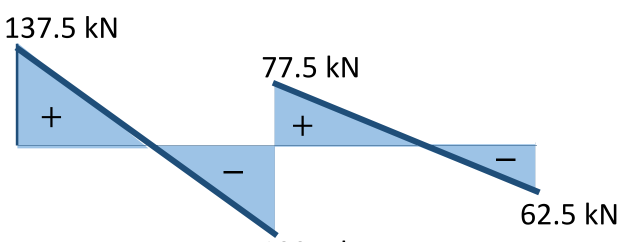
Find the moment, as follows:

$$M = 77.5x - \frac{(35)(x)^2}{2} - 66.67$$

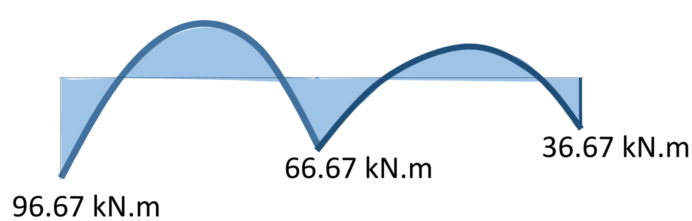
When $x = 0$, $M = -66.67$ kN.m

When $x = 4$ m, $M = -36.67$ kN.m

Shear force and bending moment diagrams.



(*e*) Shearing force diagram for the indeterminate beam



(*f*) Bending moment diagram for the indeterminate beam

Example 11.2

Using the slope-deflection method, determine the end moments and the reactions at the supports of the beam shown in Figure 11.8a, and draw the shearing force and the bending moment diagrams. *EI* = constant.

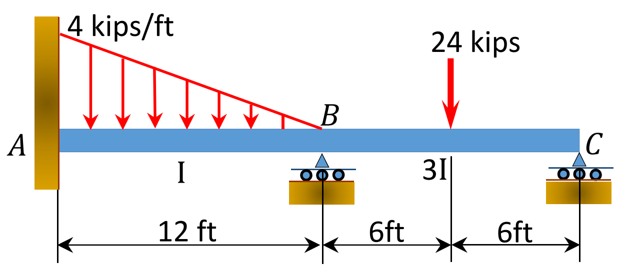


Fig. 11.8. Beam. (a)

Solution

Relative stiffness.

$$(K_{AB}):(K_{BC}) = \left(\frac{I}{12}\right):\left(\frac{3I}{12}\right) = 1:3$$

Fixed-end moments.

$$FEM_{AB} = -\frac{wL^2}{20} = -\frac{(4)(12)^2}{20} = -28.8\text{ k.ft}$$

$$FEM_{BA} = \frac{wL^2}{30} = \frac{(4)(12)^2}{30} = 19.2\text{ k.ft}$$

$$FEM_{BC} = -\frac{PL}{8} = -\frac{24\times 12}{8} = -36\text{ k.ft}$$

$$FEM_{CB} = \frac{PL}{8} = 36\text{ k.ft}$$

Slope-deflection equations.

Noting that $M_{CB} = \Psi = 0$, equations for member end moments can be expressed as follows:

$$M_{AB} = (2)(1)(2\theta_A + \theta_B - 3\psi) + FEM_{AB}$$

$$= 2\theta_B - 28.8 \qquad (1)$$

$$M_{BA} = (2)(1)(\theta_A + 2\theta_B - 3\psi) + FEM_{BA}$$

$$= 4\theta_B + 19.2 \qquad (2)$$

$$M_{BC} = 3(3)(\theta_B - \psi) + FEM_{BC} - \frac{FEM_{CB}}{2}$$

$$= 3(3)\theta_B - 36 - \frac{36}{2}$$

$$= 9\theta_B - 54 \qquad (3)$$

Joint equilibrium equation.

The equilibrium equation at joint *B* is as follows:

$$\sum M_B = M_{BA} + M_{BC} = 0$$

$$4\theta_B + 19.2 + 9\theta_B - 54 = 0$$

$$\theta_B = \frac{34.8}{13} = 2.68$$

Final end moments.

Substituting the computed value of θ_B into equations 1, 2, and 3 suggests the following:

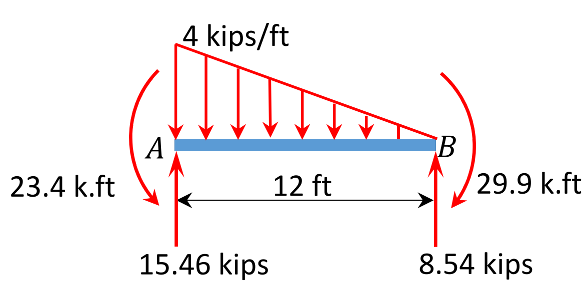
$$M_{AB} = 2(2.68) - 28.8 = -23.4 \text{ k.ft}$$

$$M_{BA} = 4\theta_B + 19.2 = 4(2.68) + 19.2 = 29.9 \text{ k.ft}$$

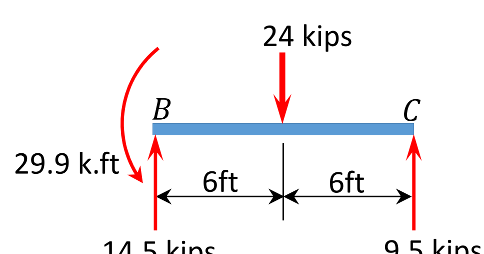
$$M_{BC} = 9\theta_B - 54 = 9(2.68) - 54 = -29.9 \text{ k.ft}$$

$$M_{CB} = 0$$

Shear force and bending moment diagrams.



(*b*)



(*c*)

Shear force and bending moment for segment *AB*.

$\curvearrowleft + \sum M_A = 0\text{: } 12B_y + 23.44 - \left(\frac{1}{2}\right)(12)(4)\left(\frac{1}{3} \times 12\right) - 29.9 = 0$

$B_y = 8.54 \text{ kips}$

$\uparrow + \sum F_y = 8.54 + A_y - \left(\frac{1}{2}\right)(12)(4) = 0$

$A_y = 15.46 \text{ kips}$

$V = -8.54 + \left(\frac{1}{2}\right)(x)\left(\frac{x}{3}\right) = -8.54 + \frac{x^2}{6}$

When $x = 0, V = -8.54 \text{ kips}$

When $x = 12 \text{ ft}, V = 15.46 \text{ kips}$

$M = 8.54x - \left(\frac{1}{2}\right)(x)\left(\frac{x}{3}\right)\left(\frac{1}{3} \times x\right) - 29.9 = 8.54x - \frac{(x)^3}{18} - 29.9$

When $x = 0, M = -29.9 \text{ k.ft}$

When $x = 12\text{ft}, M = -23.4 \text{ k.ft}$

Shear force and bending moment for segment *BC*.

$\curvearrowleft + \sum M_B = 0\text{: } 12C_y + 29.9 - (24)(6) = 0$

$C_y = 9.5 \text{ kips}$

$\uparrow + \sum F_y = 0$

$B_y + 9.5 - 24 = 0$

$B_y = 14.5 \text{ kips}$

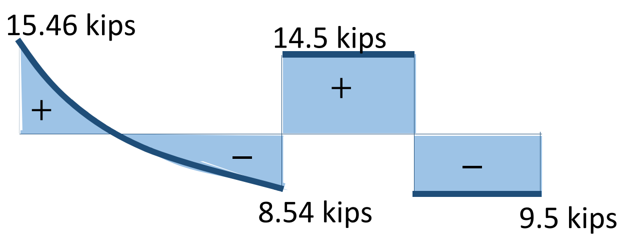
$0 < x < 6 \text{ ft}$

$V = 14.5 \text{ kips}$

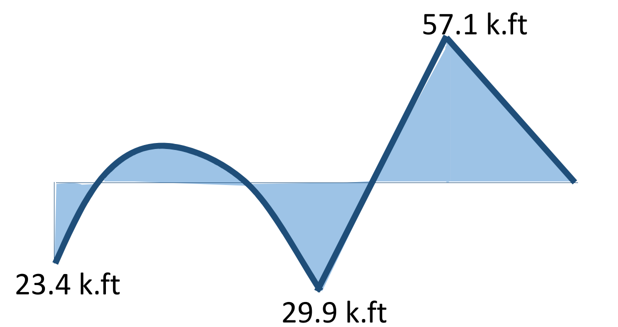
$M = 14.5x - 29.9$

When $x = 0, M = -29.9 \text{ k.ft}$

When $x = 6 \text{ ft}, M = 57.10 \text{ k.ft}$



(*d*) Shearing force for the Indeterminate beam



(*e*) Bending moment for the Indeterminate beam

Example 11.3

Using the slope-deflection method, determine the end moments of the beam shown in Figure 11.9a. Assume support *B* settles 1.5 in, and draw the shear force and the bending moment diagrams. The modulus of elasticity and the moment of inertia of the beam are 29,000 ksi and 8000 in^4, respectively.

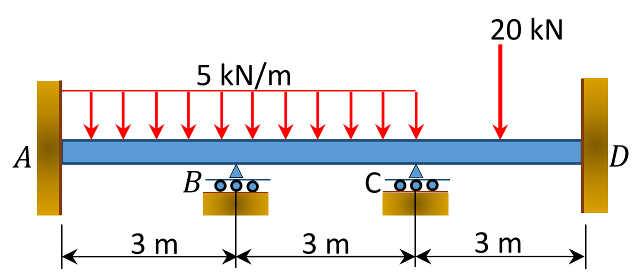


Fig. 11.9. Beam.

(*a*)

Solution

Fixed-end moments.

The Fixed-end moments (FEM) using Table 11.1 are computed as follows:

$$FEM_{AB} = -\frac{wL^2}{12} = -\frac{5\times 3^2}{12} = -3.75 \text{ kN.m}$$

$$FEM_{BA} = \frac{wL^2}{12} = 3.75 \text{ kN.m}$$

$$FEM_{BC} = -3.75 \text{ kN.m}$$

$$FEM_{CB} = 3.75 \text{ kN.m}$$

$FEM_{CD} = -\frac{\mathrm{Pab}^2}{L^2} = \frac{(20)(1.5)(1.5)^2}{3^2} = -7.5 \text{ kN.m}$

$FEM_{DC} = \frac{\mathrm{Pa}^2\mathrm{b}}{L^2} = \frac{(20)(1.5)(1.5)^2}{3^2} = 7.5 \text{ kN.m}$

Slope-deflection equations.

At $\theta_A = \theta_D = \psi = 0$, the equations for member end moments are expressed as follows:

$$M_{AB} = \frac{2\mathrm{EI}}{L}(2\theta_A + \theta_B - 3\psi) - FEM_{AB}$$

$$= 2\mathrm{EK}\theta_B - 3.75 \quad (1)$$

$$M_{BA} = \frac{2\mathrm{EI}}{L}(\theta_A + 2\theta_B - 3\psi) + FEM_{BA}$$

$$= 4\mathrm{EK}\theta_B + 3.75 \quad (2)$$

$$M_{BC} = \frac{2EI}{L}(2\theta_B + \theta_C - 3\psi) - FEM_{BC}$$

$$= 4\mathrm{EK}\theta_B + 2\mathrm{EK}\theta_C - 3.75 \quad (3)$$

$$\mathrm{M}_{CB} = \frac{2\mathrm{EI}}{L}(\theta_B + 2\theta_C - 3\psi) + \mathrm{FEM}_{CB}$$

$$= 2\mathrm{EK}\theta_B + 4\mathrm{EK}\theta_C + 3.75 \quad (4)$$

$$\mathrm{M}_{CD} = \frac{2\mathrm{EI}}{L}(2\theta_C + \theta_D - 3\psi) - \mathrm{FEM}_{CD}$$

$$= 4\mathrm{EK}\theta_C - 7.5 \quad (5)$$

$$\mathrm{M}_{DC} = \frac{2\mathrm{EI}}{L}(\theta_C + 2\theta_D - 3\psi) + \mathrm{FEM}_{DC}$$

$$= 2\mathrm{EK}\theta_C + 7.5 \quad (6)$$

Joint equilibrium equation.

The equilibrium equation at joint *B* is as follows:

$$\sum M_B = M_{BA} + M_{BC} = 0$$

$$4EK\theta_B + 3.75 + 4EK\theta_B + 2EK\theta_C - 3.75 = 0$$

$$8EK\theta_B + 2EK\theta_C = 0 \quad (7)$$

$$\sum M_C = M_{CB} + M_{CD} = 0$$

$$2EK\theta_B + 4EK\theta_C + 3.75 + 4EK\theta_C - 7.5 = 0$$

$$2EK\theta_B + 8EK\theta_C - 3.75 = 0 \qquad (8)$$

Solving equations 7 and 8 simultaneously suggests the following:

$\theta_B = -\frac{0.125}{\text{EK}}$ and $\theta_C = \frac{0.5}{\text{EK}}$

Final end moments.

Substituting the obtained values of θ_B and θ_C into the slope-deflection equations suggests the following end moments:

$$M_{AB} = 2EK\left(-\frac{0.125}{\text{EK}}\right) - 3.75 = -4.00 \text{ kN.m}$$

$$M_{BA} = 4EK\left(-\frac{0.125}{\text{EK}}\right) + 3.75 = 3.25 \text{ kN.m}$$

$$M_{BC} = 4\text{EK}\left(\frac{0.125}{\text{EK}}\right) + 2(0.5) - 3.75 = -3.25 \text{ kN.m}$$

$$M_{CB} = 2\text{EK}\left(-\frac{0.125}{\text{EK}}\right) + 4(0.5) + 3.75 = 5.50 \text{ kN.m}$$

$$M_{CD} = 4\text{EK}\left(-\frac{0.5}{\text{EK}}\right) - 7.5 = -5.50 \text{ kN.m}$$

$$M_{DC} = 2\text{EK}\left(-\frac{0.5}{\text{EK}}\right) + 7.5 = 8.5 \text{ kN.m}$$

Example 11.4

Using the slope-deflection method, determine the member end moments of the beam of the rectangular cross section shown in Figure 11.10a. Assume that support *B* settles 2 cm. The modulus of elasticity and the moment of inertia of the beam are $E = 210{,}000 \text{ N/mm}^2$ and $4.8 \times 10^4 \text{ mm},^4$ respectively.

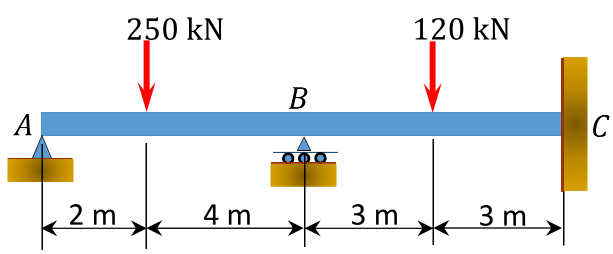


Fig. 11.10. Rectangular cross section of beam.

Solution

The Fixed-end moments (FEM) using Table 11.1 are computed as follows:

$$FEM_{AB} = -\frac{Pab^2}{L^2} = -\frac{(250)(2)(4)^2}{6^2} = -222.22 \text{ kN.m}$$

$$FEM_{BA} = \frac{Pa^2b}{L^2} = \frac{(250)(2)^2(4)}{6^2} = 111.1 \text{ kN.m}$$

$$FEM_{BC} = -\frac{PL}{8} = -\frac{(120)(6)}{8} = -90 \text{ kN.m}$$

$$FEM_{CB} = \frac{PL}{8} = \frac{(120)(6)}{8} = 90 \text{ kN.m}$$

Slope-deflection equations.

As $\theta_C = 0$, equations for member end moments are expressed as follows:

$$M_{BA} = 3EK(\theta_B - \Psi) + FEM_{BA} - \frac{FEM_{AB}}{2}$$

$$= 3EK\left(\theta_B - \frac{0.02}{6}\right) + 111.1 - \frac{(-222.2)}{2}$$

$$= 3EK\theta_B - 0.01EK + 222.2 \qquad (1)$$

$$M_{BC} = 2EK(2\theta_B + \theta_C - 3\psi) + FEM_{BC}$$

$$= 4EK\theta_B + 2EK\left(-3 \times \frac{(-0.02)}{6}\right) - 90$$

$$= 4EK\theta_B + 0.02EK - 90 \qquad (2)$$

$$M_{CB} = 2EK(\theta_B + 2\theta_C - 3\psi) + FEM_{CB}$$

$$= 2EK\theta_B + 0.02\,EK + 90 \qquad (3)$$

Joint equilibrium equation.

The equilibrium equation at joint *B* is written as follows:

$$\sum M_B = M_{BA} + M_{BC} = 0$$

$$3EK\theta_B - 0.01EK + 222.2 + 4EK\theta_B + 0.02EK - 90 = 0$$

$$7EK\theta_B + 0.01EK + 132.2 \; = \; 0 \qquad (4)$$

Solving equation 4 for θ_B suggests the following:

$$\theta_B \; = \; -0.0014 - \frac{18.89}{EK} \; = \; -0.0014 - \frac{18.89}{210 \times 10^9 K}$$

$$EK \; = \; 210 \times 10^9 \times \frac{4.8 \times 10^4}{(10^{12})(6)} = 1680$$

$$\theta_B \; = \; -0.0014 - \frac{18.89}{1680} \; = \; -0.0126 \text{ rad}$$

Final end moments.

Substituting the obtained value of θ_B into equations 1, 2, and 3 suggests the following end moments:

$M_{AB} = 0$

$$M_{BA} \; = \; 141.9 \text{ kN.m}$$

$$M_{BC} \; = \; 4EK\left(\frac{0.125}{EK}\right) + 2(0.5) - 3.75 \; = \; -141.07 \text{ kN.m}$$

$$M_{CB} \; = \; 2EK\left(-\frac{0.125}{EK}\right) + 4(0.5) + 3.75 \; = \; 81.26 \text{ kN.m}$$

Example 11.5

Using the slope-deflection method, determine the member end moments and the reactions at the supports of the frame shown in Figure 11.11a. EI = constant.

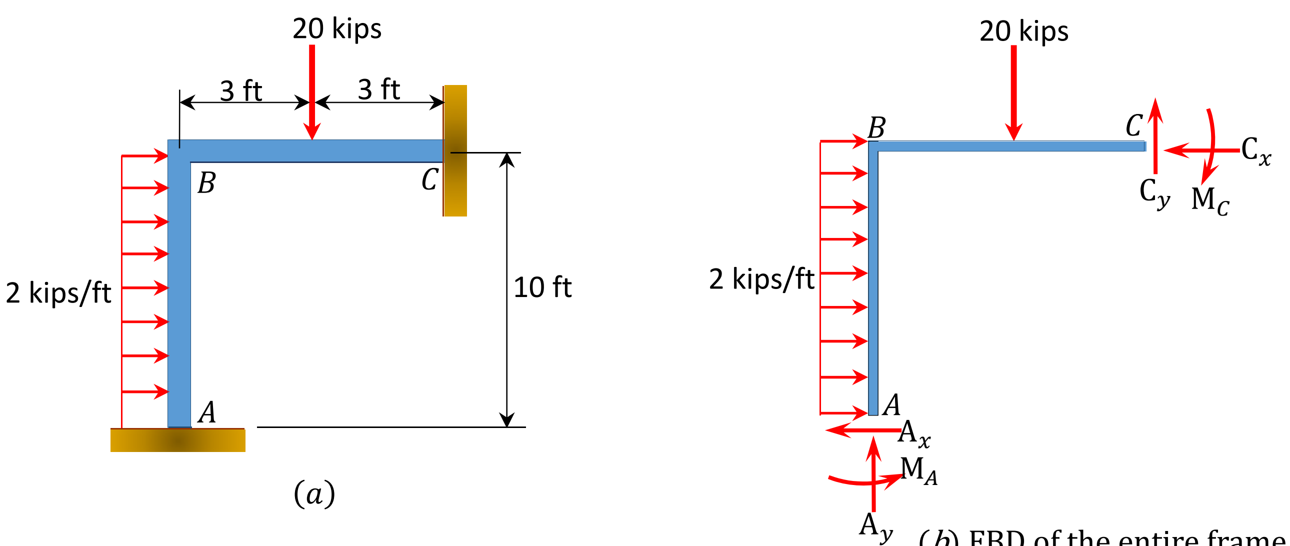
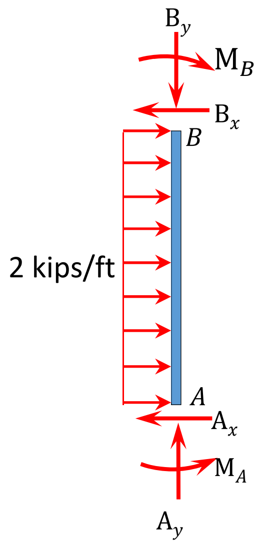
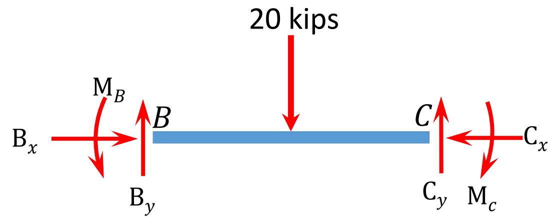


Fig. 11.11. Frame.



(c) FBD of column AB



(d) FBD of beam BC

Solution

Fixed-end moments.

The Fixed-end moments (FEM) using Table 11.1 are computed as follows:

$$FEM_{AB} = -\frac{wL^2}{12} = -\frac{2\times10^2}{12} = -16.67 \text{ k.ft}$$

$$FEM_{BA} = \frac{wL^2}{12} = 16.67 \text{ k.ft}$$

$$FEM_{BC} = -\frac{PL}{8} = -\frac{20\times6}{8} = -15$$

$$FEM_{CB} = \frac{PL}{8} = \frac{20\times6}{8} = 15$$

Slope-deflection equations.

As $\theta_A = \theta_C = 0$ due to fixity at both ends and $\psi_{AB} = \psi_{BC} = 0$ since no settlement occurs, equations for the member end moments are expressed as follows:

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3\psi) + FEM_{AB}$$

$$= 2EK\theta_B - 16.67 \qquad (1)$$

$$M_{BA} = 2EK(\theta_A + 2\theta_B - 3\psi) + FEM_{BA}$$

$$= 4EK\theta_B + 16.67 \qquad (2)$$

$$M_{BC} = 2EK(2\theta_B + \theta_C - 3\psi) + FEM_{BC}$$

$= 4EK\theta_B - 15$ (3)

$$M_{CB} = \frac{2EI}{L}(\theta_B + 2\theta_C - 3\psi) + FEM_{CB}$$

$= 2EK\theta_B + 15$ (4)

Joint equilibrium equation.

The equilibrium equation at joint B is as follows:

$$\sum M_B = M_{BA} + M_{BC} = 0$$

$$4EK\theta_B + 16.67 + 4EK\theta_B - 15 = 0$$

$$EK\theta_B = -\frac{1.67}{8} = -0.209$$

Final end moments.

Substituting $EK\theta_B = -0.209$ into equations 1, 2, 3, and 4 suggests the following:

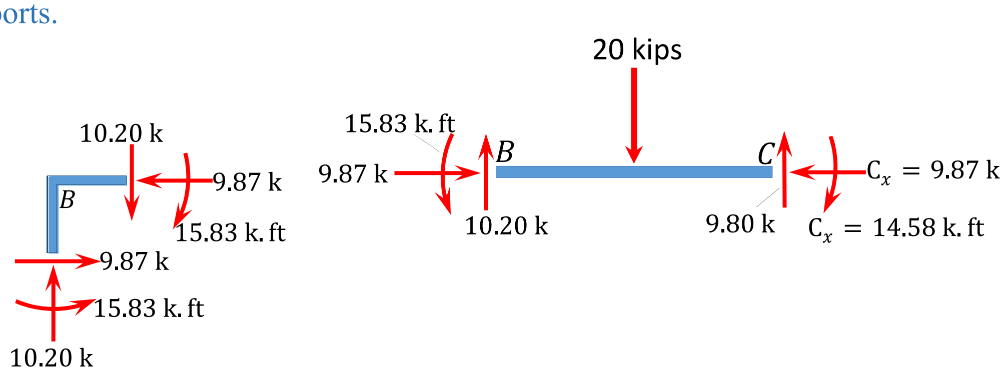
$$M_{AB} = -17.09 \text{ k.ft}$$

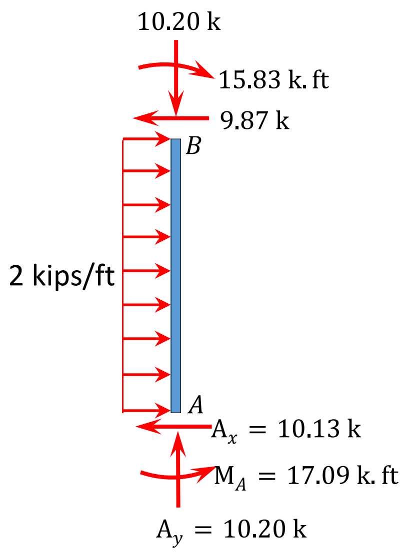
$$M_{BA} = 15.83 \text{ k.ft}$$

$$M_{BC} = -15.83 \text{ k.ft}$$

$$M_{CB} = 14.58 \text{ k.ft}$$

Reactions at supports.





(e) Member end moments, axial forces and shears

To determine A_x, take the moment about B in Figure 11.11c, as follows:

$$+\curvearrowleft \sum M_B = 0: 17.09 + (2)(10)(5) - 15.83 - 10A_x = 0$$

$$A_x = 10.13 \text{ k}$$

To determine A_y, take the moment about C in Figure 11.11b, as follows:

$$+\curvearrowleft \sum M_C = 0;\ 17.09 - 10.13 \times 10 + (2)(10)(5) + 20 \times 3 - 14.58 - 6A_y = 0$$

$$A_y = 10.20 \text{ k}$$

To determine C_y in Figure 11.11b, consider the summation of forces in the vertical direction, as follows:

$$+\uparrow \sum F_y = 0$$

$$10.20 - 20 + C_y = 0$$

$$C_y = 9.80 \text{ k}$$

To determine C_x in Figure 11.11b, consider the summation of forces in the horizontal direction, as follows:

$$+\rightarrow \sum F_x = 0$$

$$2 \times 10 - 10.13 - C_x = 0$$

$$C_x = 9.87 \text{ k}$$

Example 11.6

Using the slope-deflection method, determine the member end moments of the frame shown in Figure 11.12a.

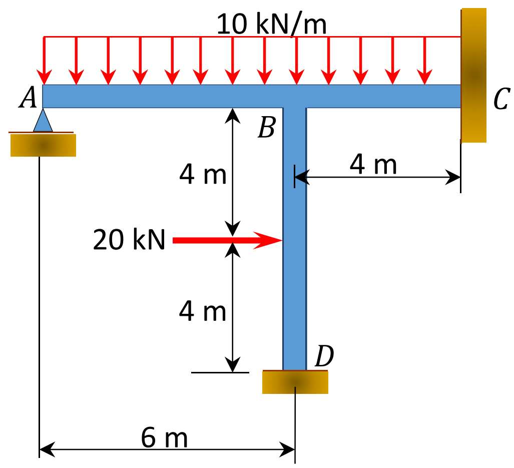


Fig. 11.12. Frame.

Solution

Fixed-end moments.

The Fixed-end moments (FEM) using Table 11.1 are computed as follows:

$$FEM_{AB} = -\frac{wL^2}{12} = -\frac{10\times6^2}{12} = -30 \text{ kN.m}$$

$$FEM_{BA} = \frac{wL^2}{12} = 30 \text{ kN.m}$$

$$FEM_{BC} = -\frac{10\times4^2}{12} = -10.33 \text{ kN.m}$$

$$\text{FEM}_{CB} = 10.33 \text{ kN.m}$$

$$FEM_{DB} = -\frac{PL}{8} = -\frac{20\times8}{8} = -20 \text{ kN.m}$$

$$FEM_{BD} = \frac{PL}{8} = \frac{20\times8}{8} = 20 \text{ kN.m}$$

Slope-deflection equations.

As $\theta_A = \theta_C = 0$ due to fixity at both ends and $\psi_{AB} = \psi_{BC} = 0$ since no settlement occurs, the equations for member end moments can be expressed as follows:

$$M_{BA} = 3EK(\theta_B - \psi) + FEM_{BA} - \frac{FEM_{AB}}{2}$$

$$= 3EK\theta_B + 30 - \frac{(-30)}{2} = 3EK\theta_B + 45 \quad (1)$$

$$M_{BC} = 2EK(2\theta_B + \theta_C - 3\psi) + FEM_{BC}$$

$$= 4EK\theta_B - 10.33 \quad (2)$$

$$M_{CB} = 2EK(\theta_B + 2\theta_C - 3\psi) + FEM_{CB}$$

$$= 2EK\theta_B + 10.33 \quad (3)$$

$$M_{DB} = 2EK(2\theta_D + \theta_B - 3\psi) + FEM_{DB}$$

$$= 2EK\theta_B - 20 \quad (4)$$

$$M_{BD} = 2EK(2\theta_B + \theta_D - 3\psi) + FEM_{BD}$$

$$= 4EK\theta_B + 20 \quad (5)$$

Joint equilibrium equation.

The equilibrium equation at joint *B* is as follows:

$$\sum M_B = M_{BA} + M_{BC} + M_{BD} = 0$$

$$3EK\theta_B + 45 + 4EK\theta_B - 10.33 + 4EK\theta_B + 20 = 0$$

$$EK\theta_B = -4.97$$

Final end moments.

Substituting $EK\theta_B = -4.97$ into equations 1, 2, 3, 4, and 5 suggests the following:

$$M_{AB} = 0$$

$$M_{BA} = 30.09 \text{ kN.m}$$

$$M_{BC} = -30.21 \text{ kN.m}$$

$$M_{CB} = 0.39 \text{ kN.m}$$

$$M_{DB} = -29.94 \text{ kN.m}$$

$M_{BD} = 0.12 \text{ kN.m}$

Example 11.7

Using the slope-deflection method, determine the member end moments of the frame shown in Figure 11.13a.

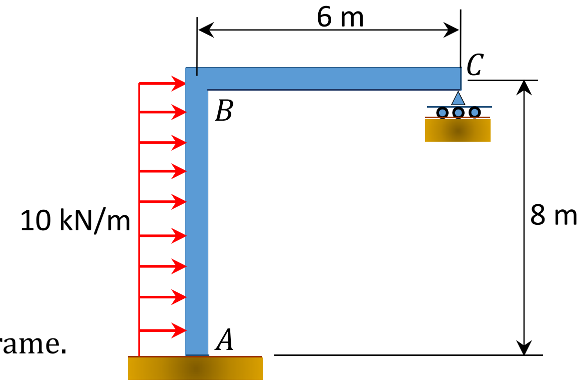


Fig. 11.13. Frame.

(a)

Solution

Fixed-end moments.

The Fixed-end moments (FEM) using Table 11.1 are computed as follows:

$$FEM_{AB} = -\frac{wL^2}{12} = -\frac{10\times8^2}{12} = -53.33 \text{ kN.m}$$

$$FEM_{BA} = \frac{wL^2}{12} = 53.33 \text{ kN.m}$$

$$FEM_{BC} = \text{FEM}_{BC} = 0$$

Slope-deflection equations.

As $\theta_A = \psi_{BC} = 0$ and $\psi_{AB} = \frac{\Delta}{8}$, the equations for member end moments can be expressed as follows:

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3\psi) + FEM_{AB}$$

$$= 2EK\left[\theta_B - 3\left(\frac{-\Delta}{8}\right)\right] - 53.33$$

$$= 2EK\theta_B + 0.75EK\Delta - 53.33 \quad (1)$$

$$M_{BA} = 2EK\left(\theta_A + 2\theta_B - 3\left(\frac{-\Delta}{8}\right)\right) + FEM_{BA}$$

$$= 2EK\left[2\theta_B - 3\left(\frac{-\Delta}{8}\right)\right] + 53.33$$

$$= 4EK\theta_B + 0.75EK\Delta + 53.33 \qquad (2)$$

$$M_{BC} = 3EK(\theta_B - \psi) + FEM_{BC} - \frac{FEM_{CB}}{2}$$

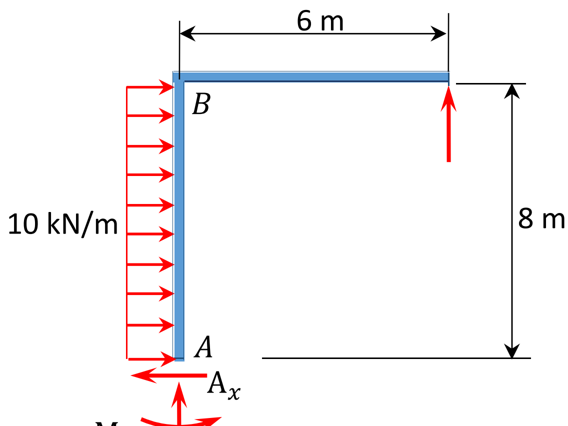
$$= 3EK\theta_B \qquad (3)$$

Joint equilibrium equation.

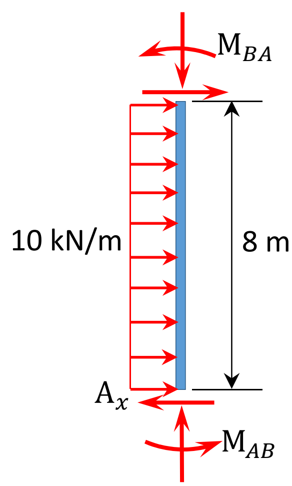
$$\sum M_B = M_{BA} + M_{BC} = 0$$

$$4EK\theta_B + 0.75EK\Delta + 53.33 + 3EK\theta_B = 0$$

$$7EK\theta_B + 0.75EK\Delta = -53..33 \qquad (4)$$



(b)



(c)

The equilibrium of the horizontal forces in Figure 11.13b suggests the following:

$$+\rightarrow \sum F_x = 0$$

$$(10)(8) - A_x = 0 \qquad (5)$$

Figure 11.13c suggests the following:

$$A_x = \frac{M_{AB} + M_{BA} + (10)(8)(4)}{8} \qquad (6)$$

Substituting A_x from equation 6 into equation 5 suggests the following:

$$80 - \frac{M_{AB}+M_{BA}+320}{8} = 0$$

$$640 - 320 = M_{AB} + M_{BA} \quad (7)$$

Substituting M_{AB} and M_{BA} from equations 1 and 2 into equation 7 suggests the following:

$$2EK\theta_B + 0.75EK\Delta - 53.33 + 4EK\theta_B + 0.75EK\Delta + 53.33 = 320$$

$$6EK\theta_B + 1.5EK\Delta = 320 \quad (8)$$

Solving equations 4 and 8 simultaneously suggests the following:

$EK\theta_B = -53.33$ and $EK\Delta = 426.66$

Final member end moments.

Putting the obtained values of $EK\theta_B$ and $EK\Delta$ into equations 1, 2, and 3 for member end moments suggests the following:

$$\mathrm{M}_{AB} = 2\mathrm{EK}\theta_B + 0.75\mathrm{EK}\Delta - 53.33 = 160\ \mathrm{kN.m}$$

$$\mathrm{M}_{BA} = 4\mathrm{EK}\theta_B + 0.75\mathrm{EK}\Delta + 53.33 = 160\ \mathrm{kN.m}$$

$$\mathrm{M}_{BC} = 3\mathrm{EK}\theta_B = -160\ \mathrm{kN.m}$$

$$\mathrm{M}_{CB} = 0$$

Example 11.8

Using the slope-deflection method, determine the member end moments of the beam of the rectangular cross section shown in Figure 11.14a.

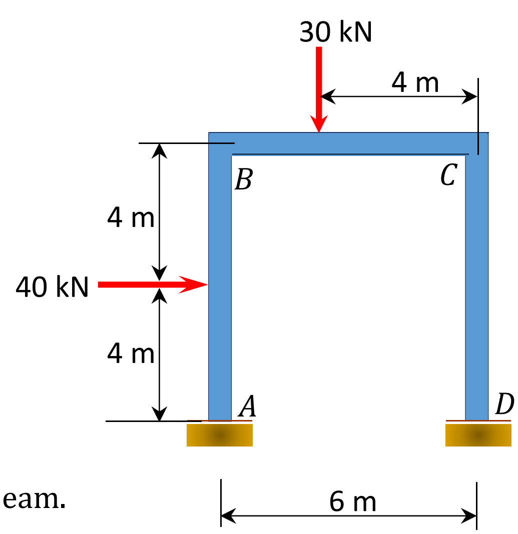


Fig. 11.14. Beam.

Solution

Fixed-end moments.

The Fixed-end moments (FEM) using Table 11.1 are computed as follows:

$$FEM_{AB} = -\frac{PL}{8} = -\frac{40\times8}{8} = -40.0\text{ kN.m}$$

$$FEM_{BA} = \frac{PL}{8} = 40.0\text{ kN.m}$$

$$FEM_{BC} = -\frac{Pab^2}{L^2} = -\frac{(30)(2)(4)^2}{6^2} = -26.67\text{ kN.m}$$

$$FEM_{CB} = \frac{Pa^2b}{L^2} = \frac{(30)(2)^2(4)}{6^2} = 13.33\text{ kN.m}$$

Slope-deflection equations.

As $\theta_A = \theta_D = 0$ and $\psi_{AB} = \frac{\Delta}{8}$, equations for member end moments can be expressed as follows:

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3\psi) + FEM_{AB}$$

$$= 2EK\left[\theta_B - 3\left(\frac{-\Delta}{8}\right)\right] - 40$$

$$= 2EK\theta_B + 0.75EK\Delta - 40 \qquad (1)$$

$$M_{BA} = 2EK\left(\theta_A + 2\theta_B - 3\left(\frac{-\Delta}{8}\right)\right) + FEM_{BA}$$

$$= 2EK\left[2\theta_B - 3\left(\frac{-\Delta}{8}\right)\right] + 40$$

$$= 4EK\theta_B + 0.75EK\Delta + 40 \qquad (2)$$

$$M_{BC} = 2EK(2\theta_B + \theta_C - 3\psi) + FEM_{BC}$$

$$= 4EK\theta_B + 2EK\theta_C - 26.67 \qquad (3)$$

$$M_{CB} = 2EK(\theta_B + 2\theta_C) + FEM_{CB}$$

$$= 2EK\theta_B + 4EK\theta_C + 13.33 \qquad (4)$$

$$M_{CD} = 2EK(2\theta_C + \theta_D - 3\psi) + FEM_{CD}$$
$$= 4EK\theta_C + 0.75EK\Delta \qquad (5)$$

$$M_{DC} = 2EK(\theta_C + 2\theta_D - 3\psi) + FEM_{DC}$$
$$= 2EK\theta_C + 0.75EK\Delta \qquad (6)$$

Joint equilibrium equation.

$$\sum M_B = M_{BA} + M_{BC} = 0$$

$$4EK\theta_B + 0.75EK\Delta + 40 + 4EK\theta_B + 2EK\theta_C - 26.67 = 0$$

$$8EK\theta_B + 2EK\theta_C + 0.75EK\Delta = -13.33 \quad (7)$$

$$\sum M_C = M_{CB} + M_{CD} = 0$$

$$2EK\theta_B + 4EK\theta_C + 13.33 + 4EK\theta_C + 0.75EK\Delta = 0$$

$$2EK\theta_B + 8EK\theta_C + 0.75EK\Delta = -13.33 \quad (8)$$

$$\sum F_x = 0$$

$$40 - A_x - D_x = 0 \quad (9)$$

Substituting $A_x = \frac{M_{AB}+M_{BA}+(40\times 4)}{8}$ and $D_x = \frac{M_{CD}+M_{DC}}{8}$ into equation 9 suggests the following:

$$40 - \frac{M_{AB} + M_{BA} + (40 \times 4)}{8} - \frac{M_{CD} + M_{DC}}{8} = 0$$

$$\frac{M_{AB} + M_{BA} + (40 \times 4)}{8} + \frac{M_{CD} + M_{DC}}{8} = 320$$

$$M_{AB} + M_{BA} + (40 \times 4) + M_{CD} + M_{DC} = 320 \quad (10)$$

Substituting the expressions of M_{AB}, M_{BA}, M_{CD} and M_{DC} from equations 1, 2, 5, and 6 into e suggests the following:

$$2EK\theta_B + 0.75EK\Delta - 40 + 4EK\theta_B + 0.75EK\Delta + 40 + 160 + 4EK\theta_C + 0.75EK\Delta + 2EK\theta_C + 0.75EK\Delta = 320$$

$$6EK\theta_B + 6EK\theta_C + 3EK\Delta = 160 \quad (11)$$

Solving equations 7, 8, and 11 simultaneously suggests the following:

$$EK\theta_B = -7.62$$

$$EK\theta_C = -7.62$$

$$EK\Delta = 83.81$$

Final member end moments.

Substituting the obtain values of $EK\theta_B, EK\theta_C$ and $EK\Delta$ into member end moment equations suggests the following:

$$M_{AB} = 2EK\theta_B + 0.75EK\Delta - 40 = 7.62$$

$$M_{BA} = 4EK\theta_B + 0.75EK\Delta + 40 = 72.39$$

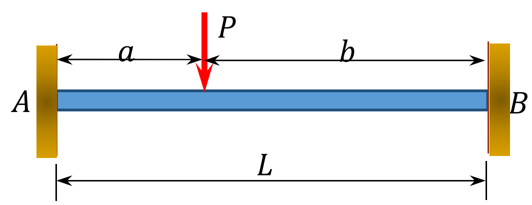
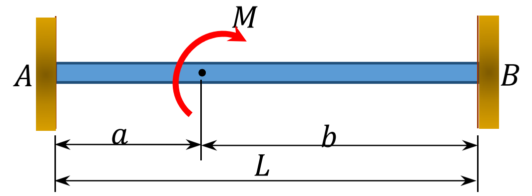
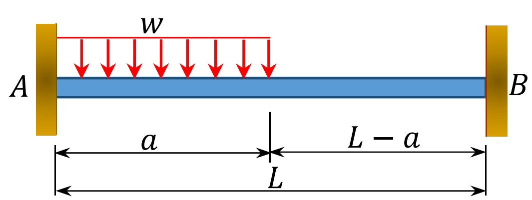
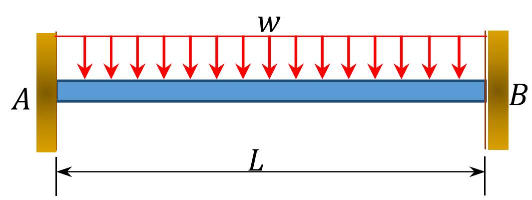
$$M_{BC} = 4EK\theta_B + 2EK\theta_C - 26.67 = -72.39$$

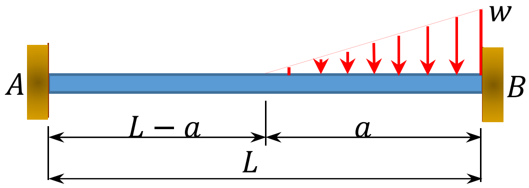
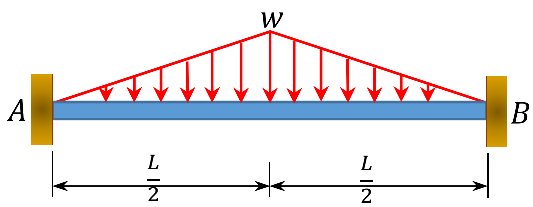
$$M_{CB} = 2EK\theta_B + 4EK\theta_C + 13.33 = -32.39$$

$$M_{CD} = 4EK\theta_C + 0.75EK\Delta = 32.39$$

$$M_{DC} = 2EK\theta_C + 0.75EK\Delta = 47.62$$

Table 11.1. Fixed-end moments.

Type of loading	$(\text{FEM})_{AB}$	$(\text{FEM})_{BA}$
	$\frac{Pab^2}{L^2}$	$\frac{Pa^2b}{L^2}$
	$b(2a-b)\frac{M}{L^2}$	$a(2b-a)\frac{M}{L^2}$
	$\frac{wL^2}{12}\left(6-8\frac{a}{L}+3\frac{a^2}{L^2}\right)$	$\frac{wL^2}{12}\left(4-3\frac{a}{L}\right)$
	$\frac{wL^2}{12}$	$\frac{wL^2}{12}$

	$\frac{wa^3}{60\,L}\left(5 - 3\frac{a}{L}\right)$	$\frac{wa^2}{60}\left(16 - 10\frac{a}{L} + 3\frac{a^2}{L^2}\right)$
	$\frac{wL^2}{30}$	$\frac{wL^2}{20}$
	$\frac{5wL^2}{96}$	$\frac{5wL^2}{96}$

Chapter Summary

Slope-deflection method of analysis of indeterminate structures: The unknowns in the slope-deflection method of analysis are the rotations and the relative displacements. Slope-deflection equations for member-end moments and the equilibrium equation at each joint that is free to rotate are written in terms of the rotations and relative displacements, and they are solved simultaneously to determine the unknowns. When the unknown rotations and the relative displacements are determined, they are put back in member end moment equations to determine the magnitude of the moments. After determination of the end moments, the structure becomes determinate. The detailed procedures for analysis by slope-deflection method for beams and frames are presented in sections 11.5 and 11.6. In situations where there are several unknowns, analysis using this method can be very cumbersome, hence the availability of software that can perform the analysis.

Slope-deflection equations for mnd Moments:

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3\psi) + M_{AB}^F$$

$$M_{BA} = 0 = 2EK(\theta_A + 2\theta_B - 3\psi) + M_{BA}^F$$

Modified slope-deflection equation when far end is supported by a roller or pin:

$$M_{AB} = 3EK(\theta_A - \psi) + \left(M_{AB}^F - \frac{M_{BA}^F}{2}\right)$$

Practice Problems

11.1 Using the slope-deflection method, compute the end moment of members of the beams shown in Figure P11.1 through Figure P11.5 and draw the bending moment and shear force diagrams. *EI* = constant.

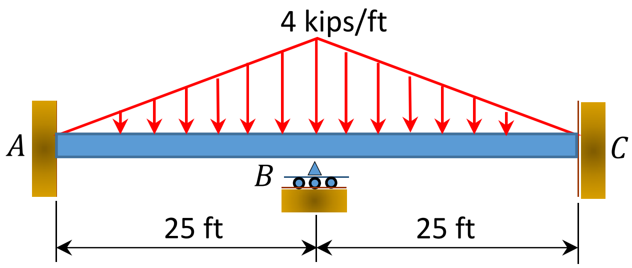


Fig. P11.1. Beam.

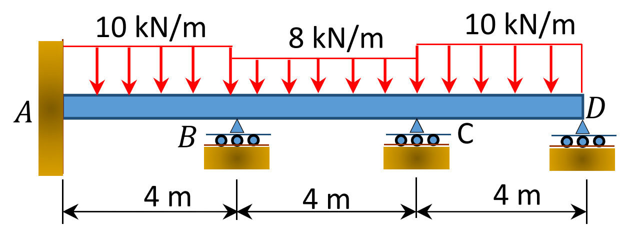


Fig. P11.2. Beam.

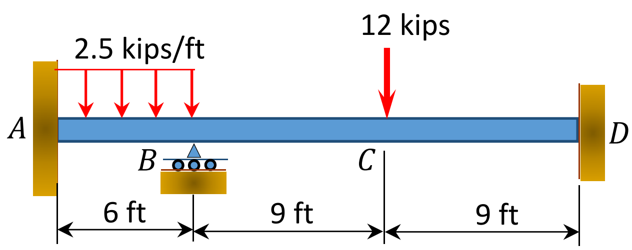


Fig. P11.3. Beam.

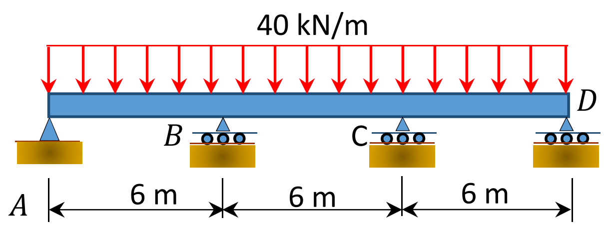


Fig. P11.4. Beam.

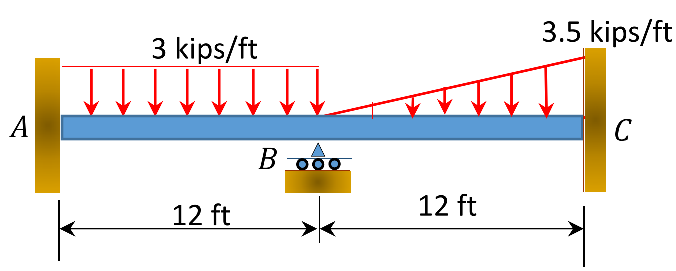


Fig. P11.5. Beam.

11.2 Using the slope-deflection method, compute the end moments of members of the beams shown in Figure P11.6. Assume support E settles by 50 mm. $E = 200$ GPa and $I = 600 \times 10^6 \text{mm}^4$.

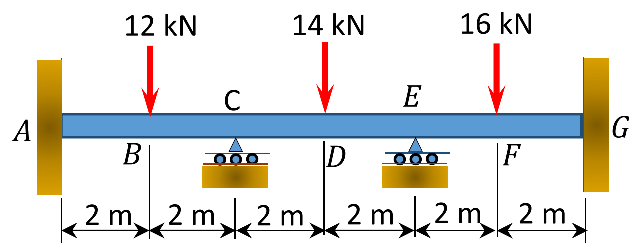


Fig. P11.6. Beam.

11.3 Using the slope-deflection method, determine the end moments of the members of the non-sway frames shown in Figure P11.7 through Figure P11.10. Draw the bending moment and the shear force diagrams.

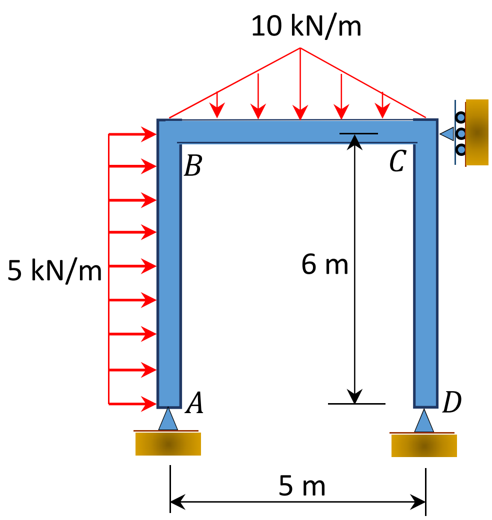


Fig. P11.7. Non-sway frame.

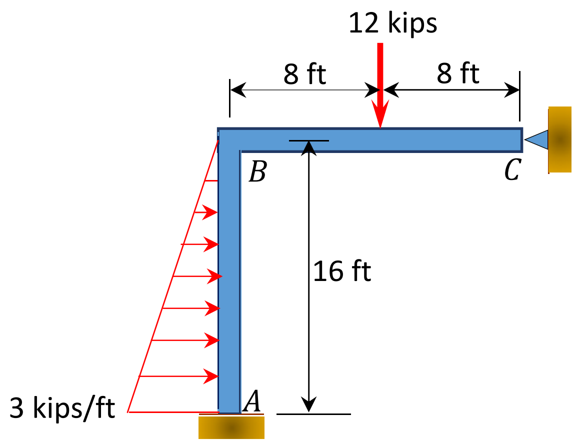


Fig. P11.8. Non – sway frame.

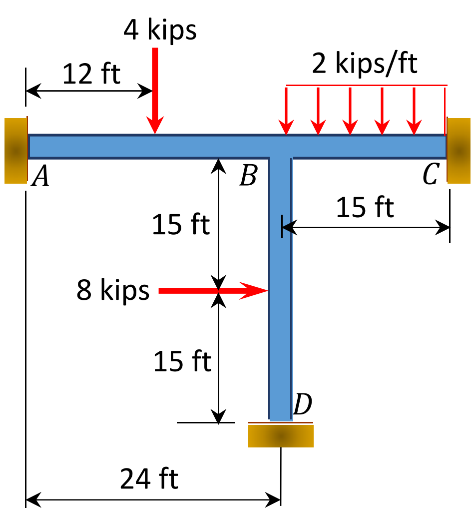


Fig. P11.9. Non – sway frame.

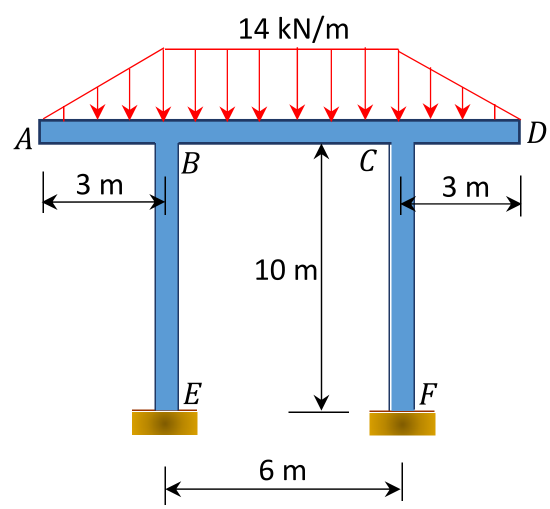


Fig. P11.10. Non – sway frame.

11.4 Using the slope-deflection method, determine the end moments of the members of the sway frames shown in Figure P11.11 through Figure P11.14. Draw the bending moment and the shear force diagrams.

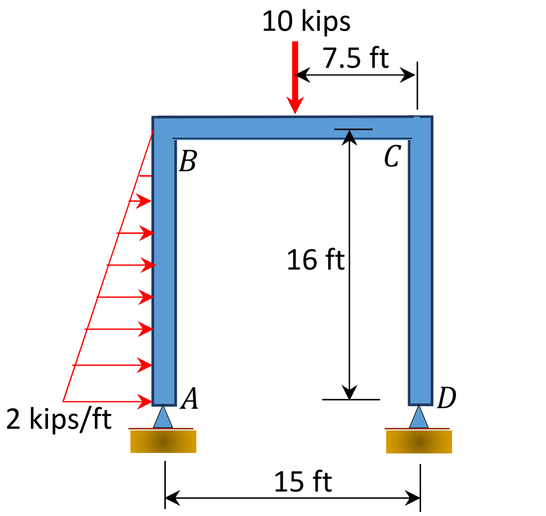


Fig. P11.11. Sway frame.

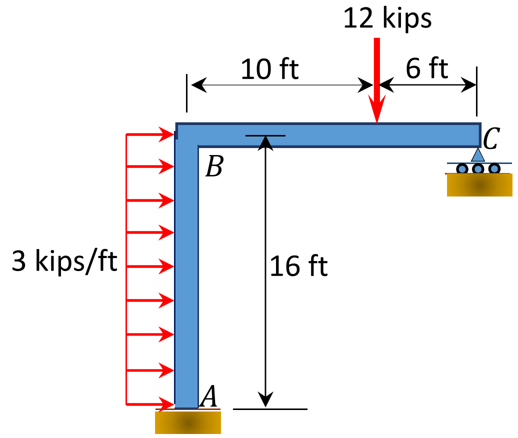


Fig. P11.12. Sway frame.

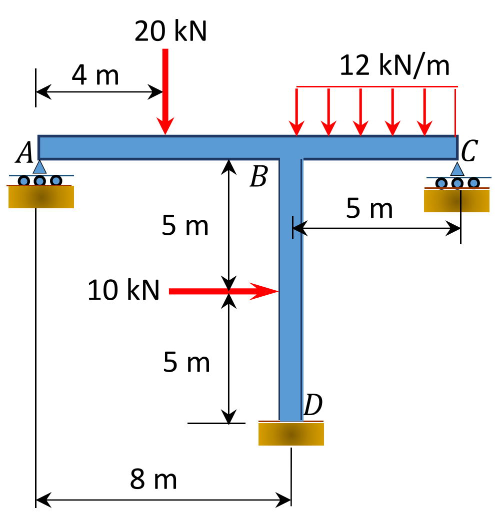


Fig. P11.13. Sway frame.

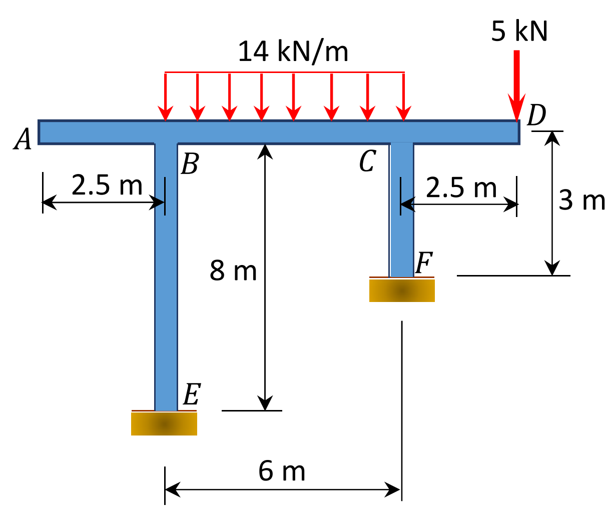


Fig. P11.14. Sway frame.

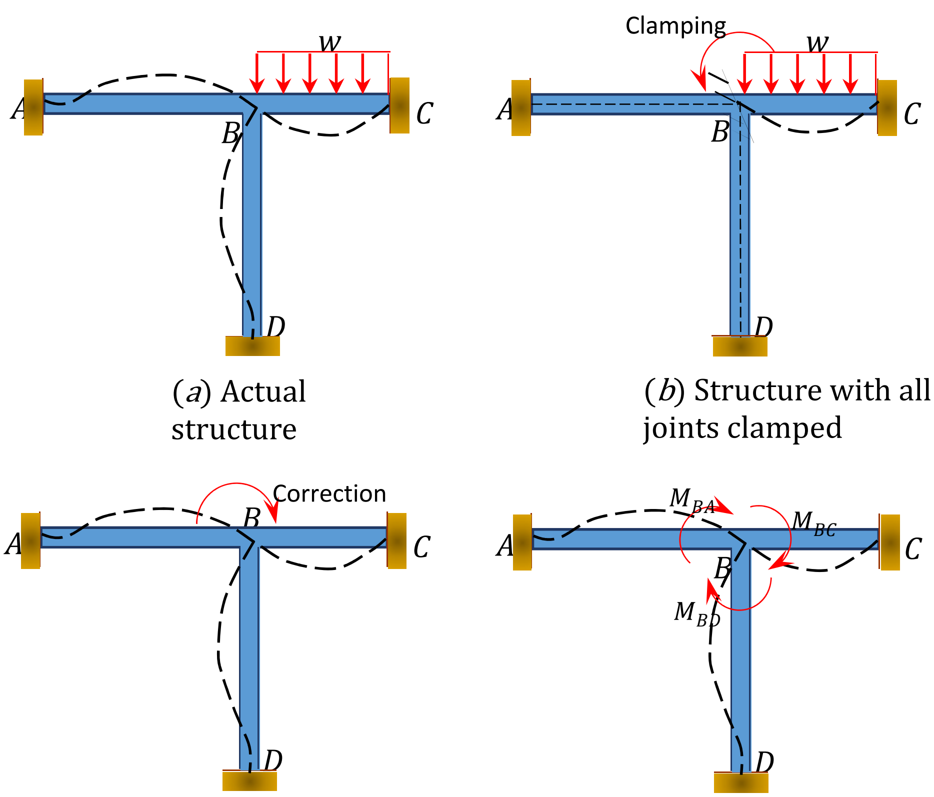
Chapter 12

Moment Distribution Method of Analysis of Structures

12.1 Basic Concepts

The moment distribution method of analysis of beams and frames was developed by Hardy Cross and formally presented in 1930. Although this method is a deformation method like the slope-deflection method, it is an approximate method and, thus, does not require solving simultaneous equations, as was the case with the latter method. The degree of accuracy of the results obtained by the method of moment distribution depends on the number of successive approximations or the iteration process.

To illustrate the concept of the method of moment distribution, consider the frame shown in Figure 12.1. Members of the frame are prismatic and are assumed not to deform axially nor translate relative to one another. Joints *ACD* of the frame are fixed, while joint *B* can rotate slightly due to the applied load. First, before carrying out moment distribution among members, all the joints are assumed to be temporarily locked using a clamp.



(*a*) Actual structure

(*b*) Structure with all joints clamped

(*c*) Structure with the clamp

(*d*) Structure with correction moment at B distributed among members

Fig. 12.1. Frame.

12.2 Sign Convention

The sign convention for the moment distribution method is similar to the one established for the slope-deflection method; that is, the moment at the end of a member is considered positive if it tends to turn the end of the member clockwise and negative if it tends to turn it counterclockwise.

12.3 Definitions

Unbalanced moments: This method of analysis assumes that the joints in a structure are initially clamped or locked and then released successively. Once a joint is released, a rotation takes place, since the sum of the fixed end moments of the members meeting at that joint is not zero. The value of the sum of the end moments obtained is the unbalanced moment at that joint.

Carry-over moments: The distributed moments in the ends of members meeting at a joint cause moments in the other ends, which are assumed to be fixed. These induced moments at the other ends are called carry-over moments.

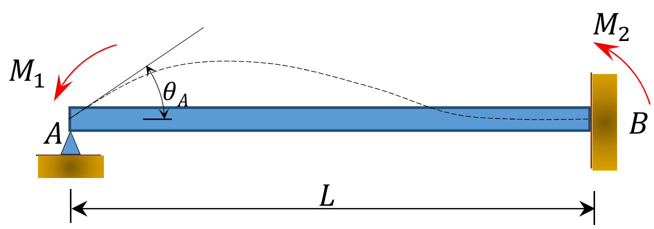


Fig. 12.2. Unloaded prismatic beam.

Consider an unloaded prismatic beam fixed at end B, as shown in Figure 12.2. If a moment M_1 is applied to the left end of the beam, the slope-deflection equations for both ends of the beam can be written as follows:

$$M_1 = 2EK(2\theta_A) = 4EK\theta_A \tag{12.1}$$

$$M_2 = 2EK\theta_A \tag{12.2}$$

Substituting $\theta_A = \frac{M_1}{4EK}$ from equation 12.1 into equation 12.2 suggest the following:

$$M_2 = \frac{1}{2}M_1 \tag{12.3}$$

Equation 12.3 suggests that the moment carried over to the fixed end of a beam due to a moment applied at the other end is equal to one-half of the applied moment.

Carry-over factor: The ratio of the induced moment to the applied moment is referred to as the carry-over factor. For the beam shown in Figure 12.2, the carry-over factor is as follows:

$$\frac{M_2}{M_1} = \frac{2EK\theta_A}{4EK\theta_A} = \frac{1}{2} \tag{12.4}$$

Distributed factor (DF): The distributed factor is a factor used to determine the proportion of the unbalanced moment carried by each of the members meeting at a joint. For the members meeting at joint O of the frame shown in Figure 12.3, their distribution factors are computed as follows:

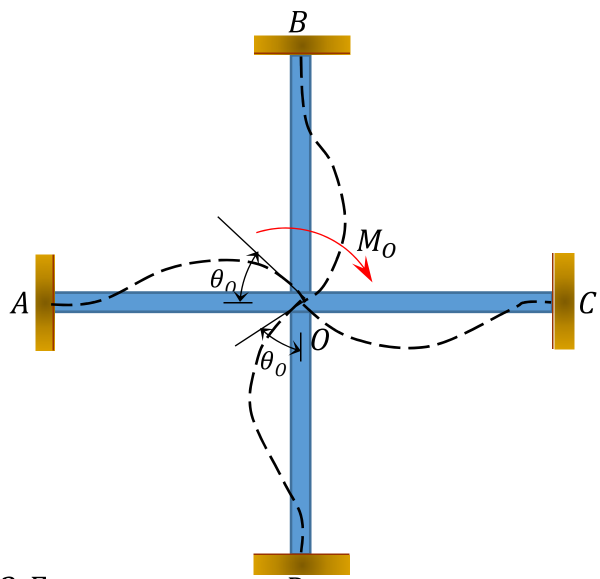


Fig. 12.3. Frame.

$$(DF)_{OA} = \frac{K_{OA}}{\sum K}$$
$$(DF)_{OB} = \frac{K_{OB}}{\sum K} \qquad (12.5)$$
$$(DF)_{OC} = \frac{K_{OC}}{\sum K}$$
$$(DF)_{OD} = \frac{K_{OD}}{\sum K}$$

Distributed moments: Upon the release of the imaginary clamp at a joint, the unbalanced moment at that joint causes it to rotate. The rotation twists the end of the members meeting at the joint, resulting in the development of resisting moments. These resisting moments are called distributed moments. The distributed moments for the members of the frame shown in Figure 12.3 are computed as follows:

$$M_{OA} = \frac{K_{OA}}{\sum K} M_O = (DF)_{OA} M_O$$
$$M_{OB} = \frac{K_{OB}}{\sum K} M_O = (DF)_{OB} M_O \qquad (12.6)$$
$$M_{OC} = \frac{K_{OC}}{\sum K} M_O = (DF)_{OC} M_O$$
$$M_{OD} = \frac{K_{OD}}{\sum K} M_O = (DF)_{OD} M_O$$

12.4 Modification of Member Stiffness

Sometimes the iteration process in the moment distribution method can be significantly reduced by adjusting the flexural stiffness of some members of the indeterminate structure. This section considers the influence of a fixed- and a pin-end support on the flexural stiffness of an indeterminate beam.

Case 1: A beam hinged at one end and fixed at the other

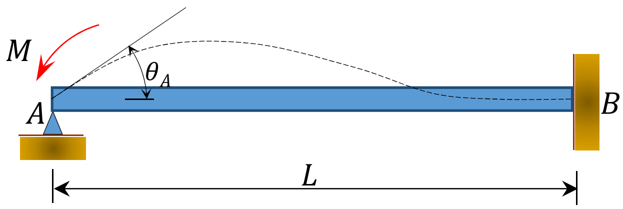


Fig. 12.4. Beam

Consider a beam hinged at end *A* and fixed at end *B*, as shown in Figure 12.4. Applying a moment *M* rotates the hinge end by an amount θ. Writing the slope-deflection equation for the end *A* of the member and noting that $\theta_B = \psi_{AB} = M_{AB}^F = 0$ suggests the following:

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi_{AB}) + M_{AB}^F$$

$$= \frac{2EI}{L}(2\theta_A + 0 - 0) + 0$$

$$M_{AB} = \left(\frac{4EI}{L}\right)\theta_A \qquad (12.7)$$

By definition, the bending stiffness of a structural member is the moment that must be applied to an end of the member to cause a unit rotation of that end. The following expression for the bending stiffness for the member with a fixed far end is expressed as follows when substituting $\theta_A = 1$ into equation 12.7:

$$K = \frac{4EI}{L} \qquad (12.8)$$

By definition, the relative bending stiffness of a member is determined by dividing the bending stiffness of the member by *4E*. Dividing the equation 12.8 by *4E* suggests the following expression for relative stiffness for the case being considered:

$$K_R = \frac{4EI}{4EL} = \frac{I}{L} \qquad (12.9)$$

Case 2: A beam hinged at both ends

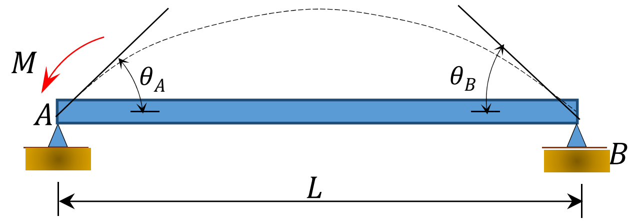


Fig. 12.5. Simply supported beam.

Applying a moment M at the end A of the simply supported beam shown in Figure 12.5 rotates the beam by an angle θ_A at the hinged end. Using the modified slope-deflection equation derived in section 11.4 of Chapter 11 and noting that $\psi = M_{AB}^F = M_{BA}^F = 0$ suggests the following expression for the moment at the hinged end where the load is applied:

$$M_{AB} = \frac{3EI}{L}(\theta_A - \psi) + \left(M_{AB}^F - \frac{M_{BA}^F}{2}\right)$$

$$= \frac{3EI}{L}(\theta_A - 0) + (0 - 0)$$

$$M_{AB} = \left(\frac{3EI}{L}\right)\theta_A \quad (12.10)$$

Substituting $\theta_A = 1$ into equation 12.10 suggests the following expression for the bending stiffness for a member with a hinged far end:

$$K = \frac{3EI}{L} \quad (12.11)$$

The relative stiffness for a member with a hinged far end is obtained by dividing equation 12.11 by *4E*, as follows:

$$K_R = \frac{3EI}{4EL} = \frac{3}{4}\left(\frac{I}{L}\right) \quad (12.12)$$

Comparing equations 12.12 and 12.9 suggests that a member with a hinged far end is three-fourth as stiff as a member with the same geometry but fixed at the far end. This established fact can substantially reduce the number of iteration when analyzing beams or frames with a hinged far end using the method of moment distribution. In such cases, the relative stiffness of the beam at the near end is first adjusted according to equation 12.12, and its distribution factor is computed with the adjusted stiffness. During the balancing operation, the near end will be balanced just once with no further carrying over of moments from or to its end.

12.5 Analysis of Indeterminate Beams

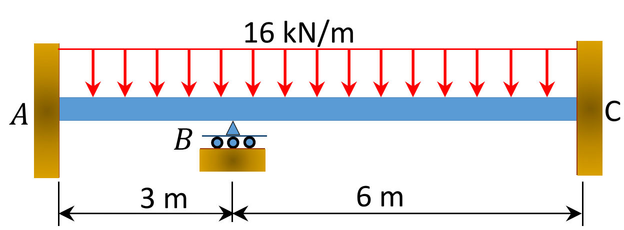
The procedure for the analysis of indeterminate beams by the method of moment distribution is briefly summarized as follows:

Procedure for Analysis of Indeterminate Beams by the Moment Distribution Method

- Calculate the fixed-end moments for members, assuming that the joints are clamped against rotation.
- Calculate the distribution factor for each of the members connected at the joint
- Calculate the unbalanced moment at each joint and distribute the same to the ends of members connected at that joint.
- Carry over one-half of the distributed moment to the other ends of members.
- Add or subtract these latter moments (moments obtained in steps three and four) to or from the original fixed-end moments.
- Apply the determined end moments at the joints of the given structure.
- Draw the free-body diagram of each span of the given beam, showing the loads and moments at the joints obtained by the moment distribution method.
- Determine the support reactions for each span.
- Compute and construct the shearing force and bending moment diagrams for each span.
- Draw one bending moment and one shearing force diagram for the given beam by combining the diagrams in step 9.

Example 12.1

Using the moment distribution method, determine the end moments and the reactions at the supports of the beam shown in Figure 12.6a. Draw the shearing force and the bending moment diagrams. EI = constant.



(a)

Fig. 12.6. Beam.

Solution

Fixed end moment.

$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{16\times3^2}{12} = -12 \text{ kN.m}$$

$$(FEM)_{BA} = \frac{wL^2}{12} = 12 \text{ kN.m}$$

$$(FEM)_{BC} = -\frac{16\times6^2}{12} = -48 \text{ kN.m}$$

$$(FEM)_{CB} = 48 \text{ kN.m}$$

Stiffness factor.

$$K_{AB} = K_{BA} = \frac{I}{3} = 0.333I$$

$$K_{BC} = K_{CB} = \frac{I}{6} = 0.167I$$

Distribution factor.

$$(DF)_{AB} = \frac{K_{AB}}{\sum K} = \frac{K_{AB}}{K_{AB}+\infty} = \frac{0.333I}{0.333I+\infty} = 0$$

$$(DF)_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA}+K_{BC}} = \frac{0.333I}{0.333I+0.167I} = 0.67$$

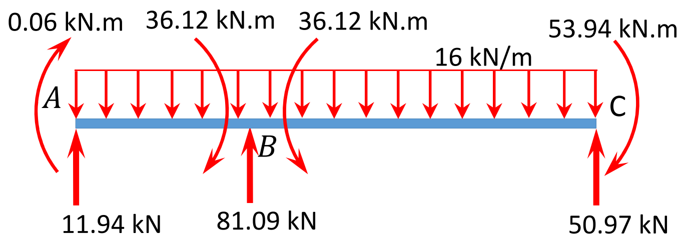
$$(DF)_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BC}}{K_{BA}+K_{BC}} = \frac{0.167I}{0.333I+0.167I} = 0.33$$

$$(DF)_{CB} = \frac{K_{CB}}{\sum K} = \frac{K_{CB}}{K_{AB}+\infty} = \frac{0.167I}{0.167I+\infty} = 0$$

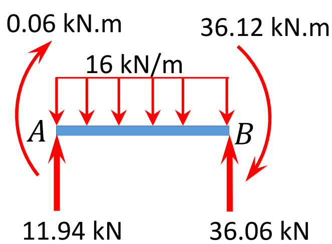
Table 12.1. Distribution table.

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.33	0.67	0
FEM Bal	-12	+12 +24.12	-48 +11.88	+48
CO	+12.06			+5.94
Total	+0.06	+36.12	-36.12	+53.94

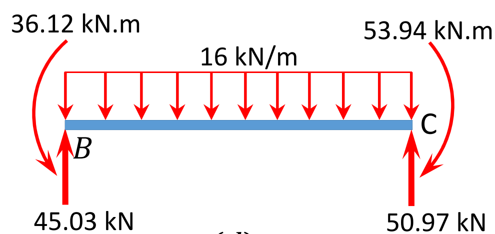
Shear force and bending moment diagrams.



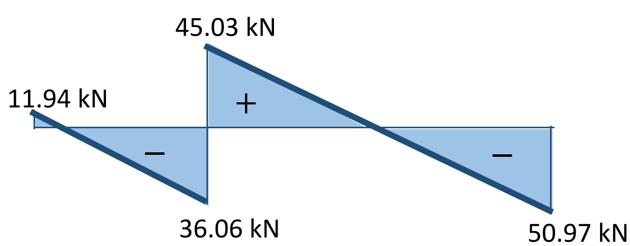
(b)



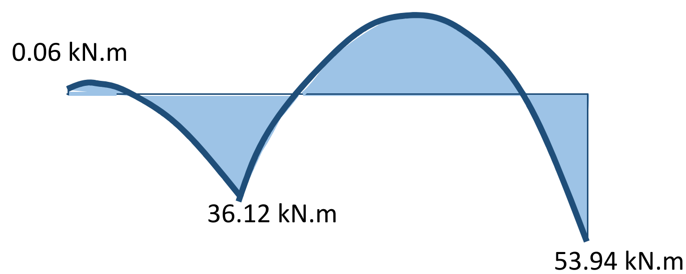
(c)



(d)



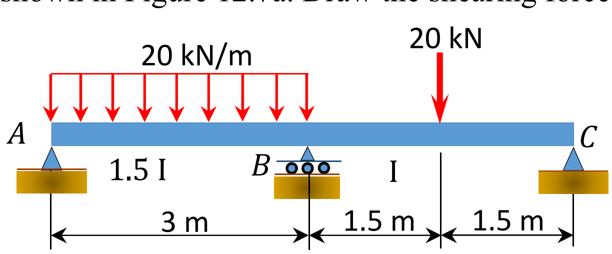
(e) Shearing force diagram for the indeterminate beam



(f) Bending moment diagram for the indeterminate beam

Example 12.2

Using the moment distribution method, determine the end moments and the reactions at the supports of the beam shown in Figure 12.7a. Draw the shearing force and the bending moment diagrams.



(a)

Fig. 12.7. Beam.

Solution

Fixed end moment.

$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{20\times3^2}{12} = -15\ \mathrm{kN.m}$$

$$(FEM)_{BA} = \frac{wL^2}{12} = +15\ \mathrm{kN.m}$$

$$(FEM)_{BC} = -\frac{PL}{8} = -\frac{20\times3}{8} = -7.5\ \mathrm{kN.m}$$

$$(FEM)_{CB} = +7.5\ \mathrm{kN.m}$$

Stiffness factor.

$$K_{AB} = K_{BA} = \frac{I_{AB}}{L_{AB}} = \frac{3}{4}\times\frac{1.5\mathrm{I}}{3} = 0.375\mathrm{I}$$

$$K_{BC} = K_{CB} = \frac{I_{BC}}{L_{BC}} = \frac{3}{4}\times\frac{\mathrm{I}}{3} = 0.25\mathrm{I}$$

Distribution factor.

$$(DF)_{AB} = \frac{K_{AB}}{\sum K} = \frac{K_{AB}}{K_{AB}+0} = \frac{0.375\mathrm{I}}{0.375\mathrm{I}+0} = 1$$

$$(DF)_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA}+K_{BC}} = \frac{0.375\mathrm{I}}{0.375\mathrm{I}+0.25\mathrm{I}} = 0.6$$

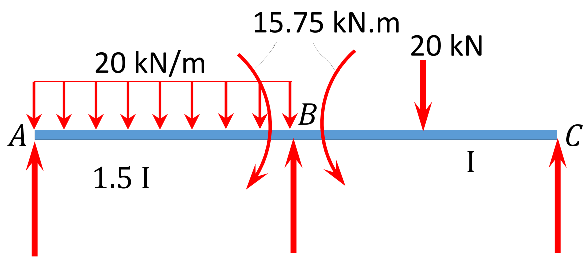
$$(DF)_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BC}}{K_{BA}+K_{BC}} = \frac{0.25\mathrm{I}}{0.375\mathrm{I}+0.25\mathrm{I}} = 0.4$$

$$(DF)_{CB} = \frac{K_{CB}}{\sum K} = \frac{K_{CB}}{K_{CB}+0} = \frac{0.25\mathrm{I}}{0.25\mathrm{I}+0} = 1$$

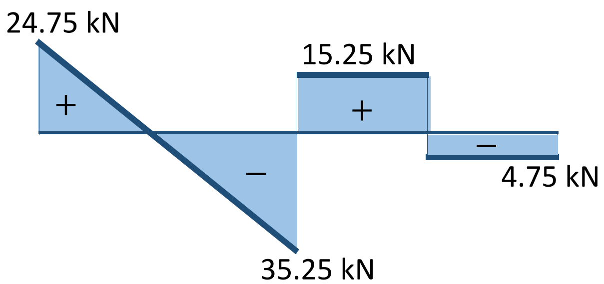
Table 12.2. Distribution table.

Joint	A	B		C
Member	AB	BA	BC	CB
DF	1	0.6	0.4	1
FEM Bal. 1	-15 +15	+15 -4.5	-7.5 -3	+7.5 -7.5
CO Bal. 2		+7.5 -2.25	-3.75 -1.5	
Total	0.0	+15.75	-15.75	0

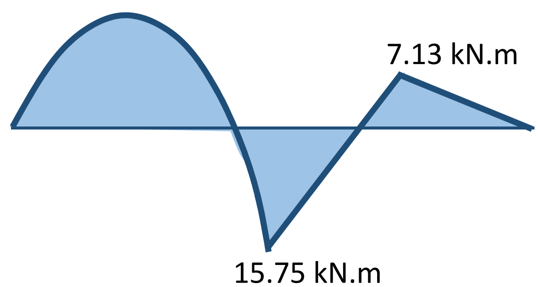
Shear force and bending moment diagrams.



(b)



(c) Shearing force diagram of the indeterminate beam



(d) Bending moment diagram of the indeterminate beam

12.6 Analysis of Indeterminate Frames

The procedure for the analysis of frames using the moment distribution method depends on the type of frame that is being analyzed. Frames are categorized as sway- or non-sway frames. The procedure for the analysis of non-sway frames are similar to that of indeterminate beams. But for the analysis of sway frames, the procedure is different. There are two stages involved in the analysis of sway frames, namely the non-sway stage and sway-stage analyses. These stages are described below.

Procedure for Analysis of Indeterminate Sway-Frames by the Moment Distribution Method

A. Non-sway stage analysis

- First assume the existence of an imaginary prop that prevents the frame from swaying.
- Compute the horizontal reactions at the supports of the frame and note the difference X. This is the force to prevent sway.

B. Sway stage analysis

- Assume arbitrary moments to act on the columns of the frame. The magnitude of these moments will vary from column to column in proportion to $\frac{I}{L^2}$.
- Values are assumed for M_2, and M_1 is determined.
- The arbitrary moments are then distributed as for the non-sway condition
- Calculate the magnitude of the horizontal reactions at the supports for the sway condition. The summation of these reactions gives the arbitrary displacing force Y.
- Determine the ratio $\frac{X}{Y}$. This ratio is called the sway factor.
- Use the sway factor to multiply the distributed moments of the sway. This gives the corrected moment for the sway.
- The final moments for the frame are the summation of the moments obtained in the non-sway stage and the corrected moment for the sway stage.

Example 12.3

Using the moment distribution method, determine the members' end moments of the frame shown in Figure 12.8. EI = constant.

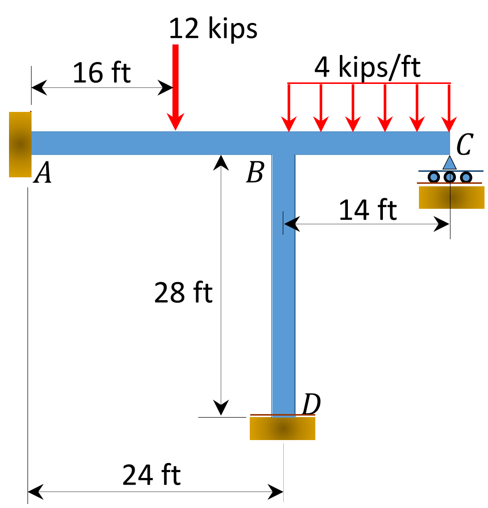


Fig. 12.8. Frame.

Solution

Fixed end moment.

$$(FEM)_{AB} = -\frac{Pab^2}{L^2} = -\frac{12\times16\times8^2}{24^2} = -21.33\ \text{k.ft}$$

$$(FEM)_{BA} = +\frac{Pa^2b}{L^2} = \frac{12\times16^2\times8}{24^2} = +42.67\ \text{k.ft}$$

$$(FEM)_{BC} = -\frac{wL^2}{12} = -\frac{4\times14^2}{12} = -65.33\ \text{k.ft}$$

$$(FEM)_{CB} = \frac{wL^2}{12} = +65.33\ \text{k.ft}$$

Stiffness factor.

$$K_{AB} = K_{BA} = \frac{I_{AB}}{L_{AB}} = \frac{\text{I}}{24} = 0.0417\text{I}$$

$$K_{BC} = K_{CB} = \frac{3}{4}\times\frac{I_{BC}}{L_{BC}} = \frac{3}{4}\times\frac{\text{I}}{14} = 0.0536\text{I}$$

$$K_{BD} = K_{DB} = \frac{I_{BD}}{L_{BD}} = \frac{I}{28} = 0.0357\text{I}$$

Distribution factor.

$$(DF)_{AB} = \frac{K_{AB}}{\sum K} = \frac{K_{AB}}{K_{AB}+0} = \frac{0.0417\text{I}}{0.0417\text{I}+\infty} = 0$$

$$(DF)_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.0417\text{I}}{0.0417\text{I}+0.0536\text{I}+0.0357\text{I}} = 0.32$$

$$(DF)_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BC}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.0536\text{I}}{0.0417\text{I}+0.0536\text{I}+0.0357\text{I}} = 0.41$$

$$(DF)_{CB} = \frac{K_{CB}}{\sum K} = \frac{K_{CB}}{K_{CB}+0} = \frac{0.0536\text{I}}{0.0536\text{I}+0} = 1$$

$$(DF)_{BD} = \frac{K_{BD}}{\sum K} = \frac{K_{BD}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.0357\text{I}}{0.0417\text{I}+0.0536\text{I}+0.0357\text{I}} = 0.27$$

$$(DF)_{DB} = \frac{K_{DB}}{\sum K} = \frac{0.0357\text{I}}{0.0357\text{I}+\infty} = 0$$

Table 12.3. Distribution table.

Joint	A	B			C	D
Member	AB	BA	BC	BD	CB	DB
DF	0	0.32	0.41	0.27	1	0
FEM Dist. 1	-21.33	+42.67 +7.25	-65.33 +9.29	0 +6.12	+65.33 −65.33	
CO Dist. 2	+3.625	 +10.453	−32.665 +13.393	 +8.82	+4.645 −4.645	+3.06
	+5.23					+4.41
Total	−12.48	+60.37	−75.31	+14.94	0.0	+7.47

Final member end moments.

Substituting the obtained values of $EK\theta_B, EK\theta_C,$ and $EK\Delta$ into the member end moment equations suggests the following:

$M_{AB} = -12.48\text{ k.ft}$

$M_{BA} = +60.37\text{ k.ft}$

$M_{BC} = -75.31\text{ k.ft}$

$$M_{BD} = +14.94\text{ k.ft}$$

$M_{CB} = 0$

$M_{DB} = +7.47\text{ k.ft}$

Example 12.4

Using the moment distribution method, determine the end moments at the supports of the frame shown in Figure 12.9. *EI* = constant.

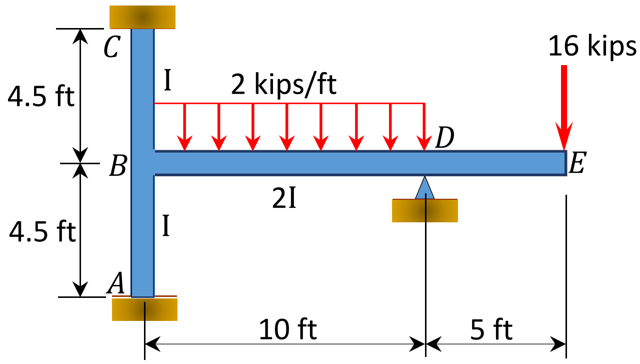


Fig. 12.9. Frame.

Solution

Fixed end moment.

$$(FEM)_{AB} = (FEM)_{BA} = (FEM)_{BC} = (FEM)_{CB} = 0$$

$$(FEM)_{BD} = -\frac{wL^2}{12} = -\frac{2\times10^2}{12} = -16.67 \text{ k.ft}$$

$$(FEM)_{DB} = \frac{wL^2}{12} = +16.67 \text{ k.ft}$$

Stiffness factor.

$$K_{AB} = K_{BA} = \frac{I_{AB}}{L_{AB}} = \frac{I}{4.5} = 0.222I$$

$$K_{BC} = K_{CB} = \frac{I_{BC}}{L_{BC}} = \frac{I}{4.5} = 0.222I$$

$$K_{BD} = K_{DB} = \frac{3}{4}\times\frac{I_{BD}}{L_{BD}} = \frac{3}{4}\times\frac{2I}{10} = 0.15I$$

Distribution factor.

$$(DF)_{AB} = 0$$

$$(DF)_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.222I}{0.222I+0.222I+0.15I} = 0.37$$

$$(DF)_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BA}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.222I}{0.222I+0.222I+0.15I} = 0.37$$

$$(DF)_{CB} = 0$$

$$(DF)_{BD} = \frac{K_{BD}}{\sum K} = \frac{K_{BD}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.15I}{0.222I+0.222I+0.15I} = 0.25$$

$$(DF)_{DB} = \frac{K_{DB}}{\sum K} = \frac{K_{DB}}{K_{DB}+0} = \frac{0.15\text{I}}{0.15\text{I}+0} = 1$$

Table 12.4. Distribution table.

Joint	A	B			C	D	E
Member	AB	BA	BC	BD	CB	DB	DE
DF	0	0.37	0.37	0.25	0	1	
CM FEM Dist. 1		 +6.17	 +6.17	 -16.67 +4.17		 +16.67 +63.33	−80
CO Dist. 2	+3.09	 −11.72	 −11.72	+31.67 −7.92	+3.09		
CO	−5.86				−5.86		
Total	−2.77	−5.55	−5.55	+11.25	−2.77	+80	−80

Final member end moments.

$M_{AB} = -2.77\ \text{k.ft}$

$M_{BA} = -5.55\ \text{k.ft}$

$M_{BC} = -5.55\ \text{k.ft}$

$M_{BD} = +11.25\ \text{k.ft}$

$$M_{CB} = -2.77$$

$M_{DB} = +80\ \text{k.ft}$

$M_{DE} = -80\ \text{k.ft}$

Example 12.5

Using the moment distribution method, determine the end moments at the supports of the frame shown in Figure 12.10. *EI* = constant.

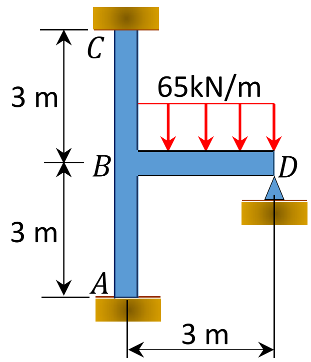


Fig. 12.10. Frame.

Solution

Fixed end moment.

$$(FEM)_{AB} = (FEM)_{BA} = (FEM)_{BC} = (FEM)_{CB} = 0$$

$$(FEM)_{BD} = -\frac{wL^2}{12} = -\frac{65\times3^2}{12} = -48.75 \text{ kN.m}$$

$$(FEM)_{DB} = \frac{wL^2}{12} = +48.75 \text{ kN.m}$$

Stiffness factor.

$$K_{AB} = K_{BA} = \frac{I_{AB}}{L_{AB}} = \frac{\text{I}}{3} = 0.333\text{I}$$

$$K_{BC} = K_{CB} = \frac{I_{BC}}{L_{BC}} = \frac{\text{I}}{3} = 0.333\text{I}$$

$$K_{BD} = K_{DB} = \frac{3}{4}\times\frac{I_{BD}}{L_{BD}} = \frac{3}{4}\times\frac{\text{I}}{3} = 0.25\text{I}$$

Distribution factor.

$$(DF)_{AB} = 0$$

$$(DF)_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.333\text{I}}{0.333\text{I}+0.333\text{I}+0.25\text{I}} = 0.36$$

$$(DF)_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BA}}{K_{BA} + K_{BC} + K_{BD}} = \frac{0.333I}{0.333I+0.333I+0.25I} = 0.36$$

$$(DF)_{CB} = 0$$

$$(DF)_{BD} = \frac{K_{BD}}{\sum K} = \frac{K_{BD}}{K_{BA} + K_{BC} + K_{BD}} = \frac{0.25I}{0.333I+0.333I+0.25I} = 0.27$$

$$(DF)_{DB} = \frac{K_{DB}}{\sum K} = \frac{K_{DB}}{K_{DB}+0} = \frac{0.25I}{0.25I+0} = 1$$

Table 12.5. Distribution table.

Joint	A	B			C	D
Member	AB	BA	BC	BD	CB	DB
DF	0	0.36	0.36	0.27	0	1
FEM				+48.75		−48.75
Dist. 1		−17.55	−17.55	−13.16		+48.75
CO	−8.78			+24.38	−8.78	
Dist. 2		−8.78	−8.78	−6.58		
CO	−4.39				−4.39	
Total	−13.17	−26.33	−26.33	+53.39	−13.17	0

Final member end moments.

$M_{AB} = -13.17\ \text{k.ft}$

$M_{BA} = -26.33\ \text{k.ft}$

$M_{BC} = -26.33\ \text{k.ft}$

$M_{BD} = +53.39\ \text{k.ft}$

$M_{CB} = -13.17\ \text{k.ft}$

$M_{DB} = 0$

Example 12.6

Using the moment distribution method, determine the member end-moments of the frame with side-sway shown in Figure 12.11a.

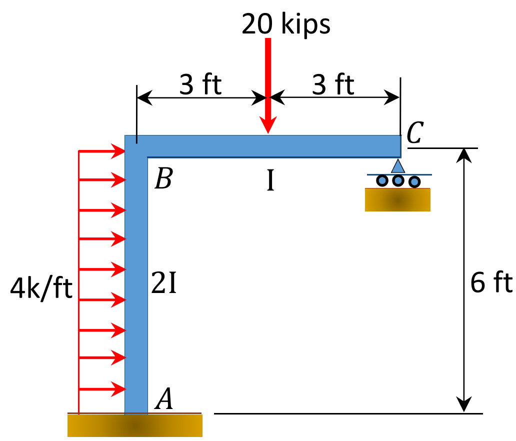


Fig. 12.11. Frame with side – sway. (a)

Solution

Fixed end moment.

$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{4\times6^2}{12} = -12\text{ k.ft}$$

$$(FEM)_{BA} = \frac{wL^2}{12} = +12\text{ k.ft}$$

$$(FEM)_{BC} = -\frac{PL}{8} = -\frac{20\times6}{8} = -15\text{ k.ft}$$

$$(FEM)_{CB} = +15\text{ k.ft}$$

Stiffness factor.

$$K_{AB} = K_{BA} = \frac{I_{AB}}{L_{AB}} = \frac{2I}{6} = 0.333I$$

$$K_{BC} = K_{CB} = \frac{I_{BC}}{L_{BC}} = \frac{3}{4}\times\frac{I}{6} = 0.125I$$

Distribution factor.

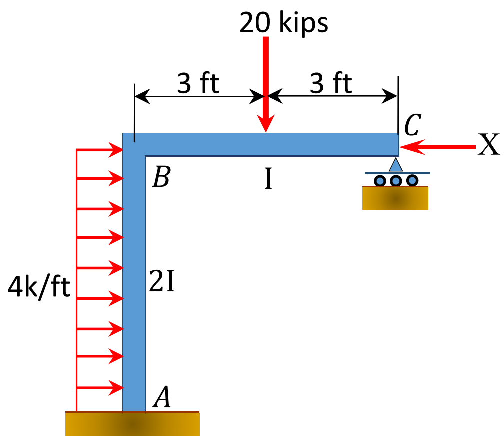
$$(DF)_{AB} = \frac{K_{AB}}{\sum K} = \frac{K_{AB}}{K_{AB}+\infty} = \frac{0.333I}{0.333I+\infty} = 0$$

$$(DF)_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{0.333\text{I}}{0.333\text{I}+0.125\text{I}} = 0.73$$

$$(DF)_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{0.125\text{I}}{0.333\text{I}+0.125\text{I}} = 0.27$$

$$(DF)_{CB} = \frac{K_{CB}}{\sum K} = \frac{K_{CB}}{K_{CB}+0} = \frac{0.125\text{I}}{0.125\text{I}+0} = 1$$

Analysis of frame without side-sway.



(b)

Table 12.6. Distribution table (no sway frame).

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.73	0.27	1
FEM Bal. 1	−12	+12 + 2.19	−15 + 0.81	+15 −15
CO Bal. 2	+1.095	 +5.475	−7.5 +2.025	
CO	+2.738			
Total	−8.17	+19.67	−19.67	0

$$\sum M_B = 0$$

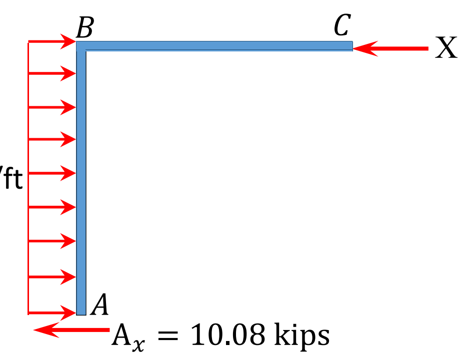
$$8.17 + (4)(6)(3) - 19.67 - 6A_x = 0$$

$$A_x = \frac{[8.17+(4)(6)(3)-19.67]}{6} = 10.08 \text{ kips}$$

$$\sum F_x = 0$$

$$(4)(6) - 10.08 - X = 0$$

$$X = 13.92 \text{ kips}$$



(d)

Analysis of frame with side-sway.

Assume that $M_{AB} = +20$ k.ft

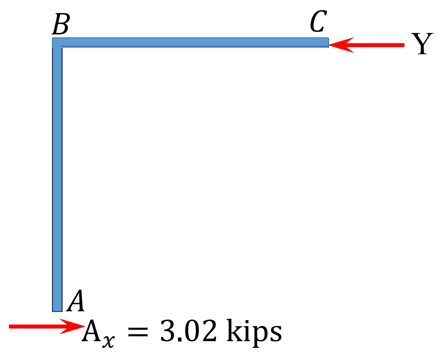
Table 12.7. Distribution table (sway frame).

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.73	0.27	1
FEM	+20	+20		
Bal. 1		− 14.6	− 5.4	
CO	− 7.3			
Total	+12.7	+5.4	− 5.4	0

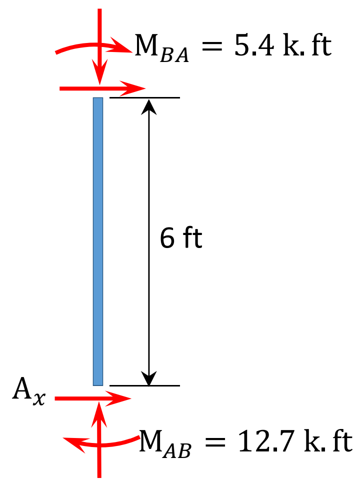
$$\sum M_B = 0$$

$$-12.7 - 5.4 + 6A_x = 0$$

$$A_x = \frac{(12.7+5.4)}{6} = 3.02 \text{ kips}$$



(e)



(f)

$$\sum F_x = 0$$

$$3.02 - Y = 0$$

$$Y = 3.02 \text{ kips}$$

Corrective factor $\eta = \frac{X}{Y} = \frac{13.92}{3.02} = 4.61$

Final end moments.

$$M_{AB} = -8.17 + (12.7)(4.61) = 50.38 \text{ k.ft}$$

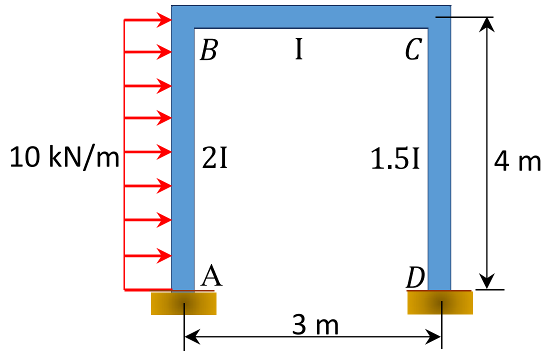
$$M_{BA} = 19.67 + (5.4)(4.61) = 44.56 \text{ k.ft}$$

$$M_{BC} = -19.67 + (-5.4)(4.61) = -44.56 \text{ k.ft}$$

$$M_{CB} = 0$$

Example 12.7

A sway frame is loaded as shown in Figure 12.12a. Using the moment distribution method, determine the end moments of the members of the frame.



(a)

Fig. 12.12. Loaded sway frame.

Solution

Fixed end moment.

$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{10\times4^2}{12} = -13.33 \text{ kN.m}$$

$$(FEM)_{BA} = \frac{wL^2}{12} = +13.33 \text{ kN.m}$$

Stiffness factor.

$$K_{AB} = K_{BA} = \frac{I_{AB}}{L_{AB}} = \frac{2I}{4} = 0.5I$$

$$K_{BC} = K_{CB} = \frac{I_{BC}}{L_{BC}} = \frac{I}{3} = 0.333I$$

$$K_{CD} = K_{DC} = \frac{I_{BD}}{L_{BD}} = \frac{1.5I}{4} = 0.375I$$

Distribution factor.

$$(DF)_{AB} = \frac{K_{AB}}{\sum K} = \frac{K_{AB}}{K_{AB}+0} = \frac{0.5I}{0.5I+\infty} = 0$$

$$(DF)_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA}+K_{BC}} = \frac{0.5I}{0.5I+0.333I} = 0.60$$

$$(DF)_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BC}}{K_{BA}+K_{BC}} = \frac{0.333I}{0.333I+0.5} = 0.40$$

$$(DF)_{CB} = \frac{K_{CB}}{\sum K} = \frac{K_{CB}}{K_{CB}+K_{CD}} = \frac{0.333I}{0.333I+0.375I} = 0.47$$

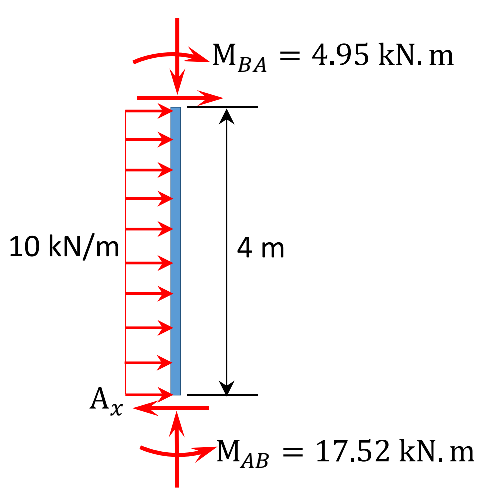
$$(DF)_{CD} = \frac{K_{CD}}{\sum K} = \frac{K_{CD}}{K_{CB}+K_{CD}} = \frac{0.375I}{0.333I+0.375I} = 0.53$$

$$(DF)_{DC} = \frac{K_{DC}}{\sum K} = \frac{0.375I}{0.375I+\infty} = 0$$

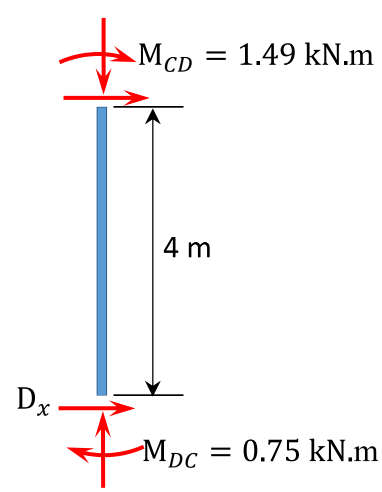
Analysis of frame without side-sway.

Table 12.8. Distribution table (no sway frame).

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.60	0.4	0.47	0.53	0
FEM	−13.33	+13.33				
Dist. 1		−8.00	−5.33			
CO	−4.00			−2.67		
Dist. 2				+1.25	+1.42	
CO			+0.63			+0.71
Dist. 3		−0.38	−0.25			
CO	−0.19			−0.13		
Dist. 4				+0.06	+0.07	
CO						+0.04
Total	−17.52	+4.95	−4.95	−1.49	+1.49	+0.75



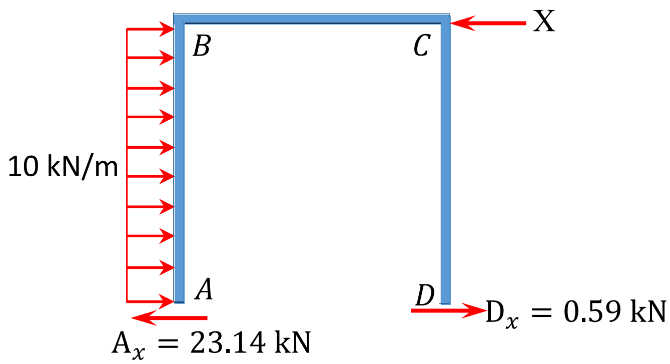
(b)



(c)

$$A_x = \frac{[17.52 - 4.95 + (10)(4)(2)]}{4} = 23.14\text{kN}$$

$$D_x = \frac{1.49 + 0.75}{4} = 0.59 \text{ kN}$$



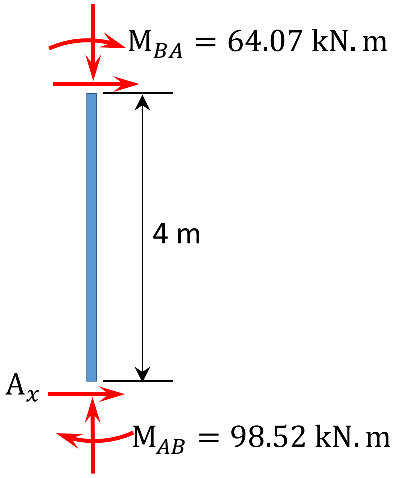
(*d*)

$$X = (10)(4) + 0.59 - 23.14 = 17.45 \text{ kN}$$

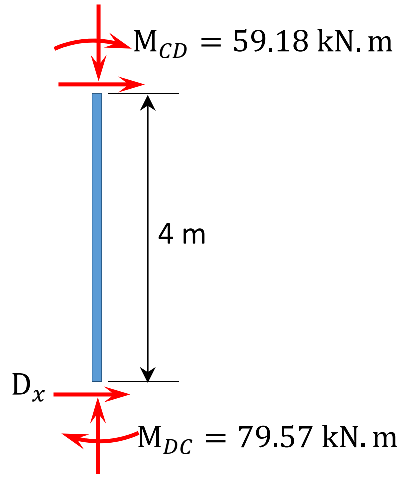
Table 12.9. Distribution table (sway frame).

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.6	0.40	0.47	0.53	0
FEM	+133	+133			+100	+100
Dist. 1		−79.8	−53.2	−47.0	−53.0	
CO	−39.9		−23.5	−26.6		−26.5
Dist. 2		+14.1	+9.40	+12.50	+14.10	
CO	+7.05		+6.25	+4.7		+7.05
Dist. 3		−3.75	−2.50	−2.21	−2.49	
CO	−1.88		−1.11	−1.25		−1.25
Dist. 4		+0.67	+0.44	+0.59	+0.66	
CO	+0.34		+0.30	+0.22		+0.33
Dist. 5		−0.18	−0.12	−0.10	−0.12	
CO	−0.09		−0.05	−0.06		−0.06
Dist. 6		+0.03	+0.02	+0.03	+0.03	
Total	+98.52	+64.07	−64.07	−59.18	+59.18	+79.57

Analysis of frame with side-sway.



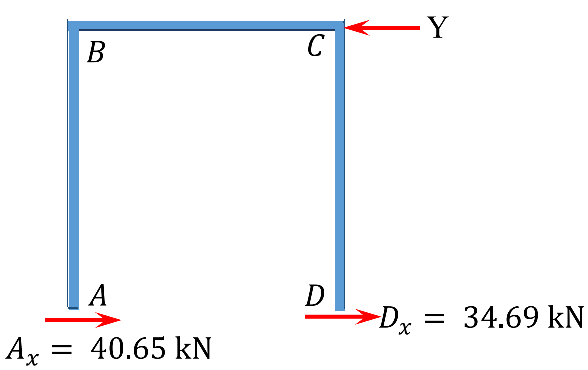
(e)



(f)

$$A_x = \frac{98.52+64.07}{4} = 40.65 \text{ kN}$$

$$D_x = \frac{79.57+59.18}{4} = 34.69 \text{ kN}$$



(g)

$$Y = 40.65 + 34.69 = 75.34 \text{ kN}$$

$$\eta = \frac{X}{Y} = \frac{17.45}{75.34} = 0.23$$

Final end moment.

$$M_{AB} = -17.52 + (98.52)(0.23) = 5.14 \text{ kN.m}$$

$M_{BA} = 4.95 + (64.07)(0.23) = 19.69\ \text{kN.m}$

$M_{BC} = -4.95 + (-64.07)(0.23) = -19.69\ \text{kN.m}$

$M_{CB} = -1.49 + (-59.18)(0.23) = -15.10\ \text{kN.m}$

$M_{CD} = 1.49 + (59.18)(0.23) = 15.10\ \text{kN.m}$

$M_{DC} = 0.75 + (79.57)(0.23) = 19.05\ \text{kN.m}$

Chapter Summary

Moment distribution method of analysis of indeterminate structures: The moment distribution method of analysis is an approximate method of analysis. Its degree of accuracy is dependent on the number of iterations. In this method, it is assumed that all joints in a structure are temporarily locked or clamped and, thus, are prevented from possible rotation. Loads are applied to the members, and the moments developed at the member ends due to fixity are determined. Joints in the structure are then unlocked successively, and the unbalanced moment at each joint is distributed to members meeting at that joint. Carry over moments at members' far ends are determined, and the process of balancing is continued until the desired level of accuracy. Members' end moments are determined by adding up the fixed-end moment, the distributed moment, and the carry over moment. Once members' end moments are determined, the structure becomes determinate.

Practice Problems

12.1 Use the moment distribution method to compute the end moment of members of the beams shown in Figure P12.1 through Figure P12.12 and draw the bending moment and shear force diagrams. *EI* = constant.

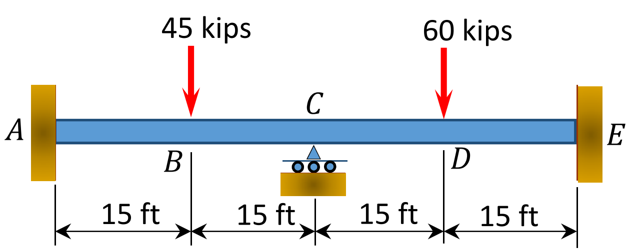


Fig. P12.1. Beam.

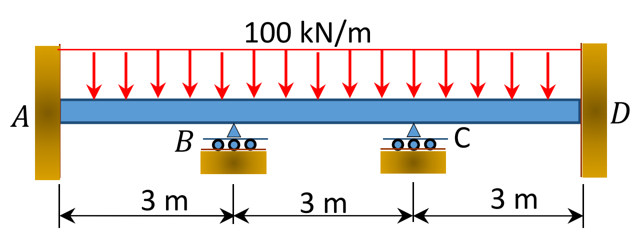


Fig. P12.2. Beam.

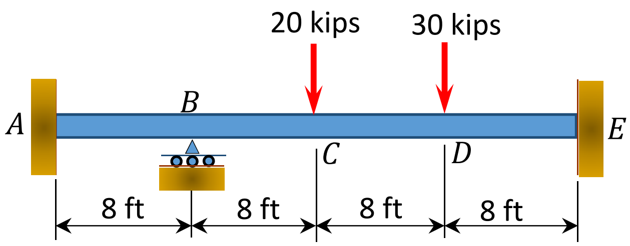


Fig. P12.3. Beam.

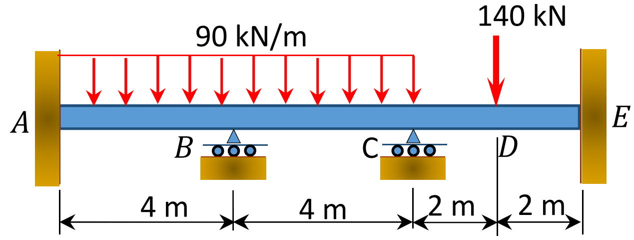


Fig. P12.4. Beam.

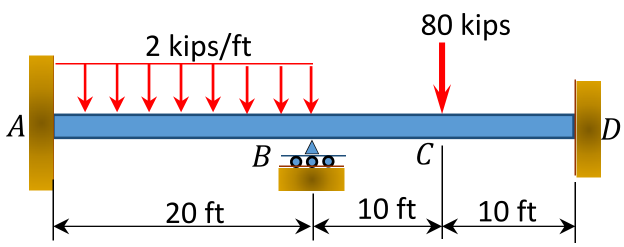


Fig. P12.5. Beam.

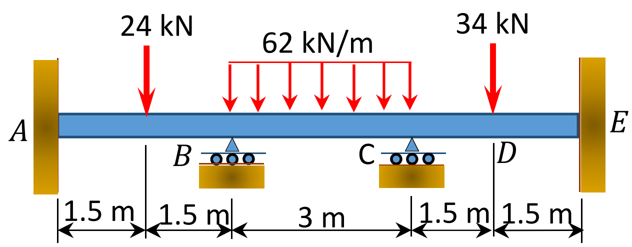


Fig. P12.6. Beam.

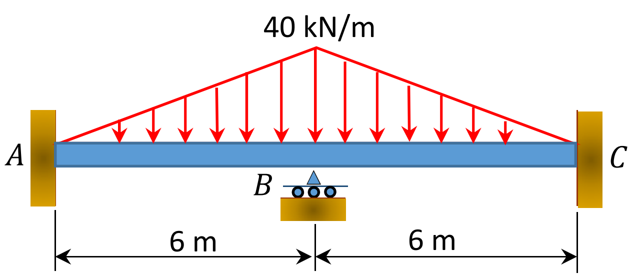


Fig. P12.7. Beam.

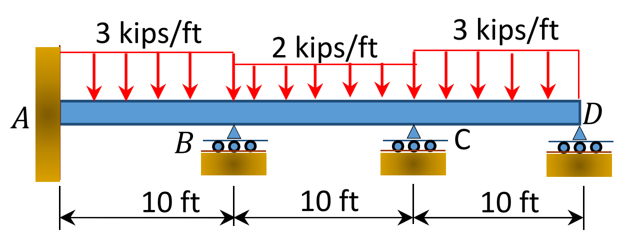


Fig. P12.8. Beam.

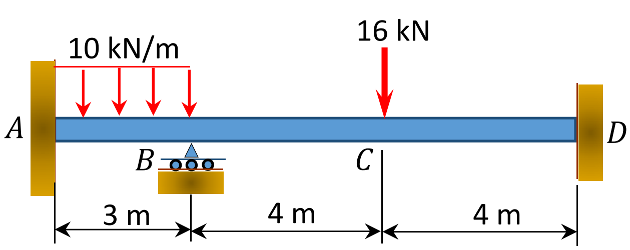


Fig. P12.9. Beam.

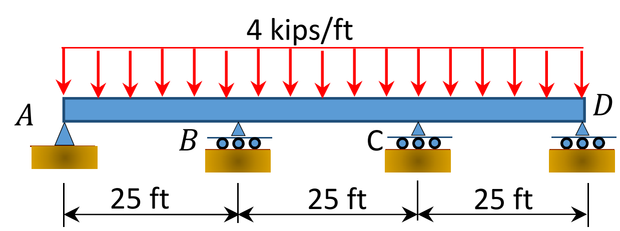


Fig. P12.10. Beam.

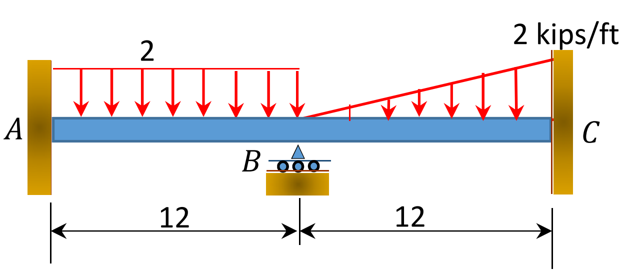


Fig. P12.11. Beam.

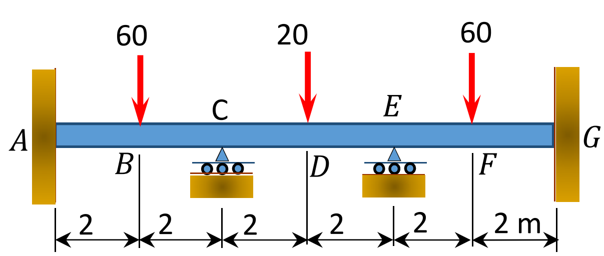


Fig. P12.12. Beam.

12.2 Use the moment distribution method to compute the end moment of the members of the frames shown in Figure P12.13 through Figure 12.20 and draw the bending moment and shear force diagrams. *EI* = constant.

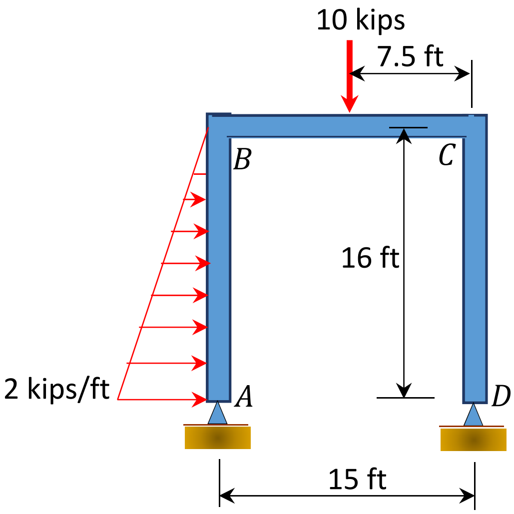


Fig. P12.13. Frame.

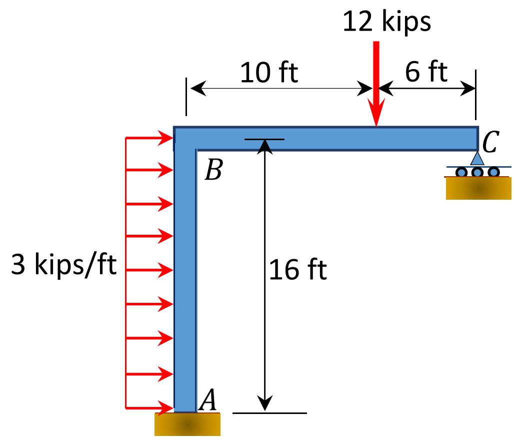


Fig. P12.14. Frame.

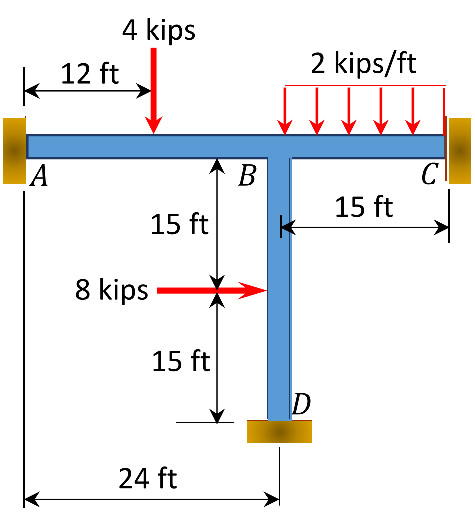


Fig. P12.15. Frame.

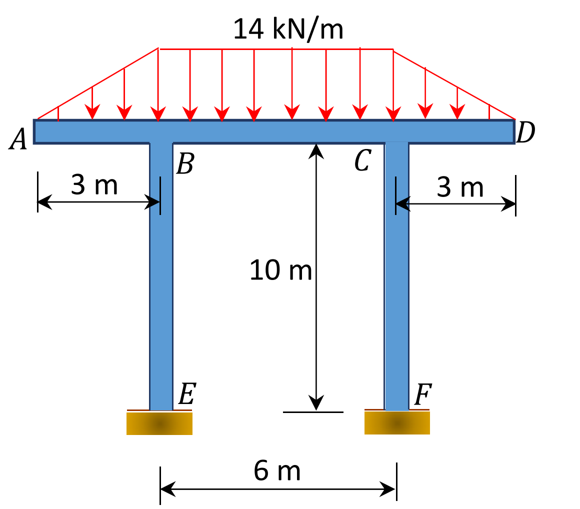


Fig. P12.16. Frame.

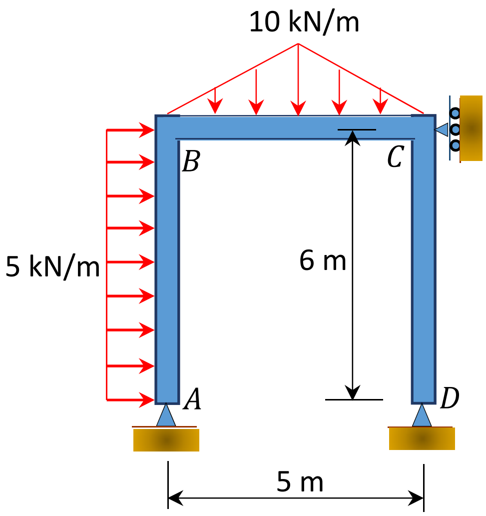


Fig. P12.17. Frame.

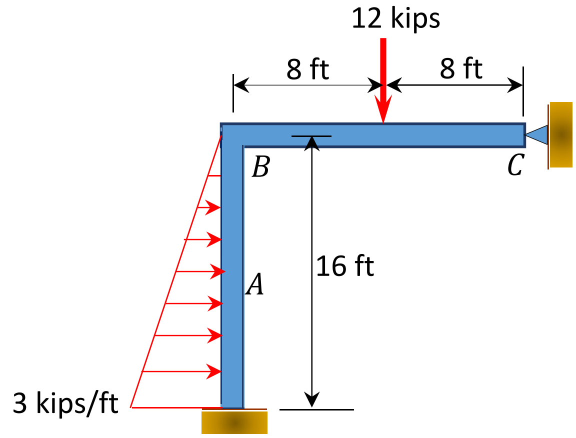


Fig. P12.18. Frame.

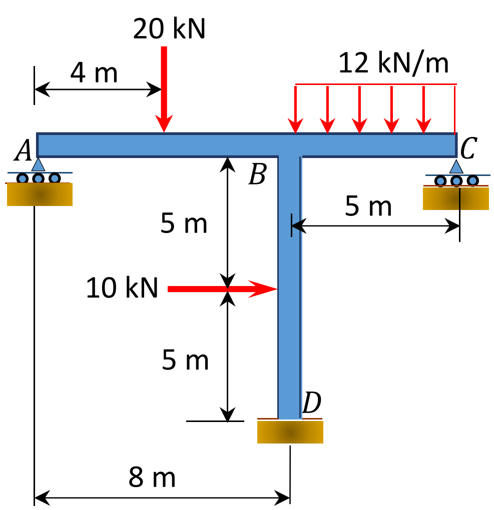


Fig. P12.19. Frame.

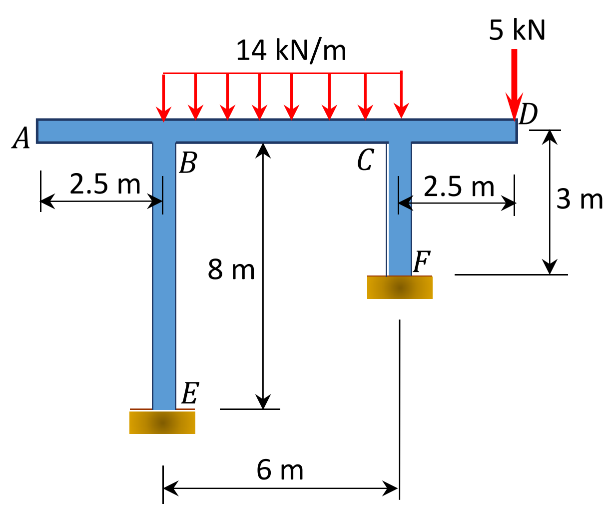


Fig. P12.20. Frame.

Chapter 13

Influence Lines for Statically Indeterminate Structures

13.1 Introduction

The influence lines for statically indeterminate structures are obtained by the static equilibrium method or by the kinematic method, as was the case for determinate structures. The procedures for finding influence lines for indeterminate structures by these methods are similar to those outlined in chapter nine for determinate structures. The distinguishing feature between the graphs of the influence lines for determinate and indeterminate structures is that the former contains straight lines while the later consists of curves. The analysis and constructions of the influence lines using the equilibrium and kinematic methods are discussed in this chapter.

13.2 Static Equilibrium Method

To construct the influence line for the reaction at the prop of the cantilever beam shown in Figure 13.1, first determine the degree of indeterminacy of the structure. For the propped cantilever, the degree of indeterminacy is one, as the beam has four reactions (three at the fixed end and one at the prop). Thus, the propped cantilever has one reaction more than the three equations of equilibrium. Considering the reaction at the prop as the redundant and removing it from the system provides the primary structure. The next step is to apply a unit load at various distances x from the fixed support and at the position where the redundant was removed. Then, compute the deflections at these points on the beam using any method. The redundant B_y at the prop can be determined using the following compatibility equation:

$$\delta_{BX} + \delta_{BB} B_y = 0$$

From which

$$B_y = -\frac{\delta_{BX}}{\delta_{BB}}$$

where

δ_{BX} = deflection at B due to the unit load at any arbitrary point on the primary structure at a distance x from the fixed support.

δ_{BB} = deflection at B due to the unit value of the redundant (i.e., $B_y = 1$).

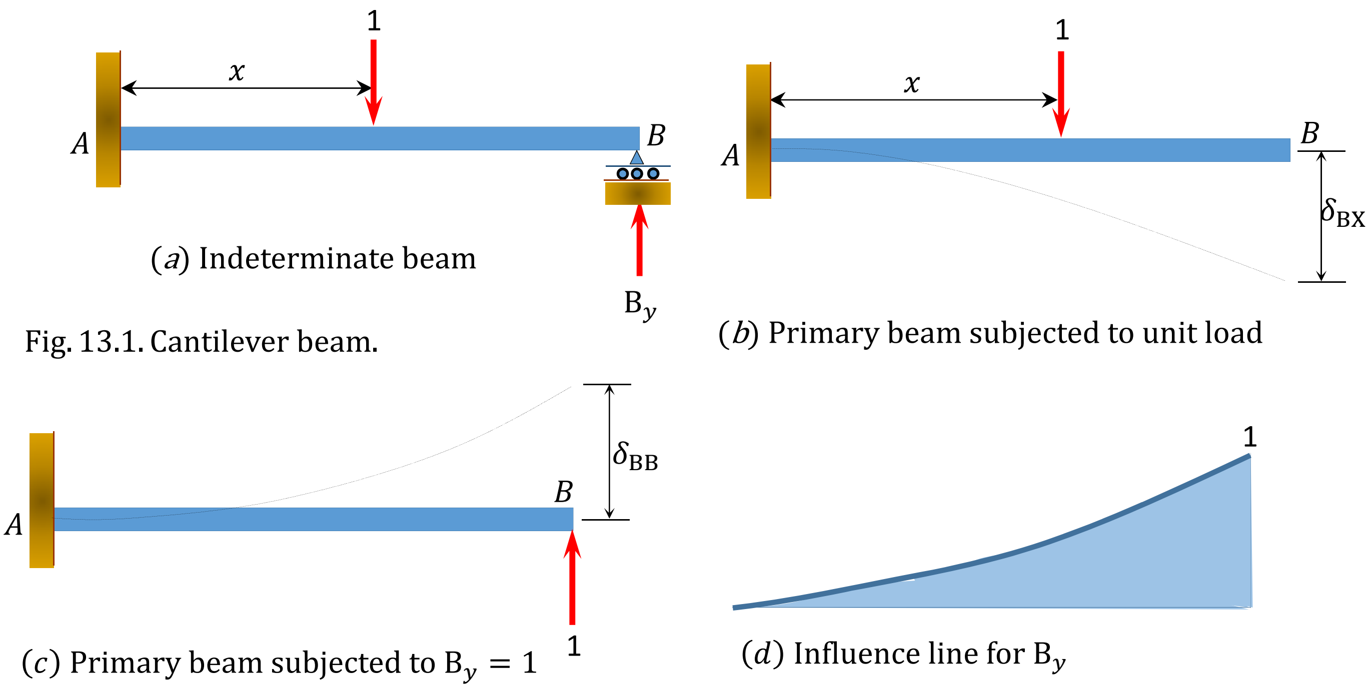


Fig. 13.1. Cantilever beam.

Example 13.1

Draw the influence lines for the reactions at supports *A* and *B* and the moment and shear force at point *C* of the propped cantilever beam shown in Figure 13.2a.

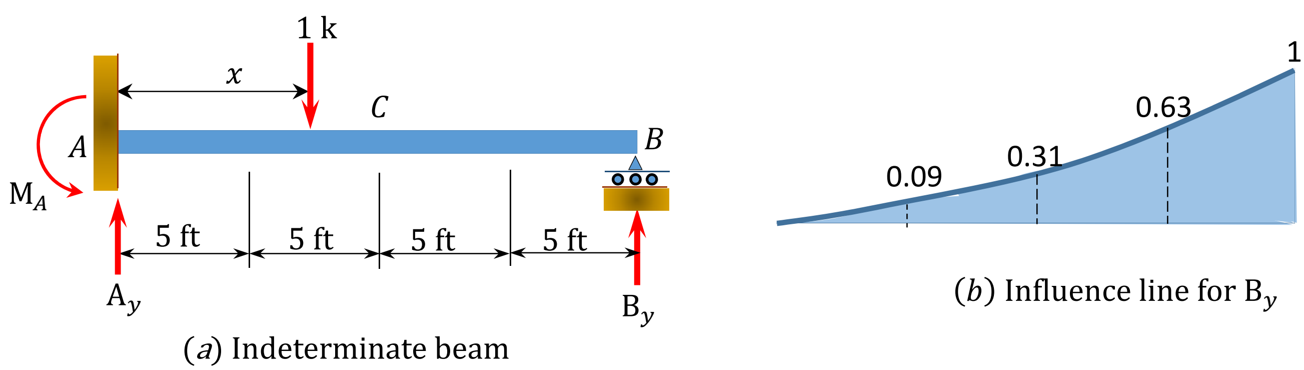
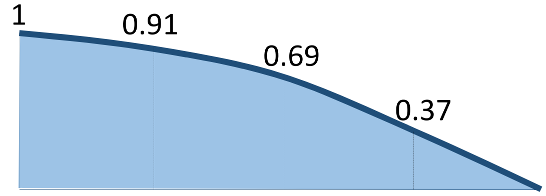
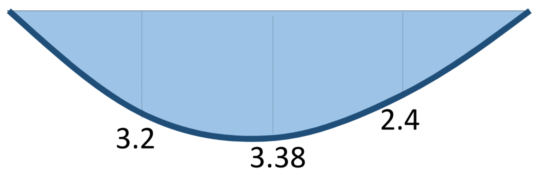


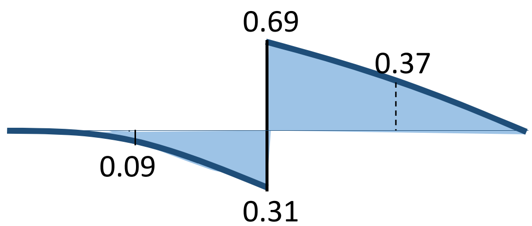
Fig. 13.2. Propped cantilever beam.



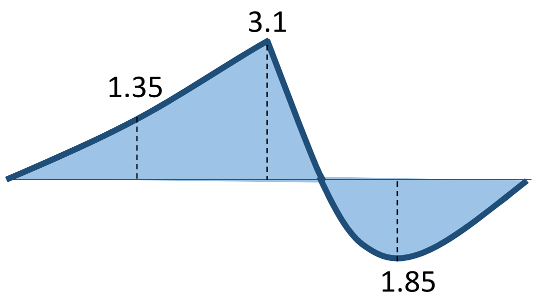
(c) Influence line for A_y



(d) Influence line for M_A



(e) Influence line for V_C



(f) Influence line for M_C

Solution

The degree of indeterminacy of the beam is one. By selecting the reaction at the prop as the redundant, the value of this redundant can be determined by solving the following compatibility equation when the unit load is located at any point *x* along the beam:

$$\delta_{BX} + \delta_{BB} B_y = 0$$

Therefore, $$B_y = -\frac{\delta_{BX}}{\delta_{BB}}$$

Using the deflection formulas provided in appendix *A* of this book, the deflections at the prop due to a unit load acting at a quarter span interval along the beam can be determined as follows:

$$\delta_{BX} = \frac{P}{6EI}(x^3 - 60x^2)$$

$$\delta_{B1} = \delta_{BA} = 0$$

$$\delta_{B2} = -229.17$$

$$\delta_{B3} = -833.33$$

$$\delta_{B4} = -1687.5$$

$$\delta_{B5} = -2666.67$$

$$\delta_{BB} = -\delta_{B5} = 2666.67$$

The ordinates of the influence lines for the desired functions are tabulated in Table 13.1

Table 13.1.

x(ft)	$(EI)\delta_B$	B_y	A_y	M_A	V_C	M_C
0	0	0	1	0	0	0
5	-229.17	0.09	0.91	-3.2	-0.09	1.35
10	-833.33	0.31	0.69	-3.8	-0.31(L) 0.69 (R)	3.1
15	-1687.5	0.63	0.37	-2.4	0.37	-1.85
20	-2666.67	1	0	0	0	0

Example 13.2

Draw the influence lines for the reactions at the supports A, B, and C of the indeterminate beam shown in Figure 13.3.

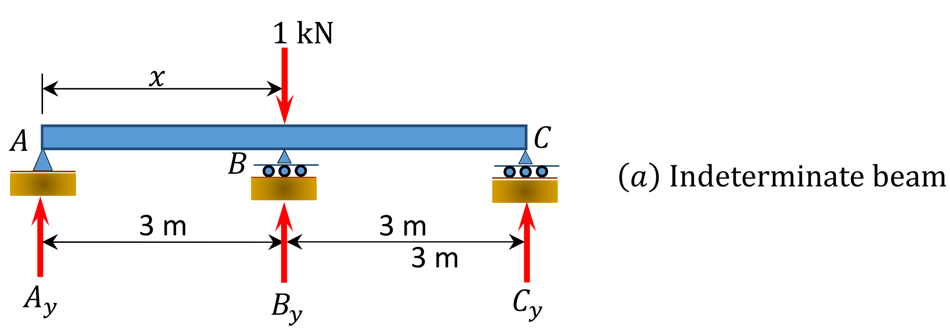
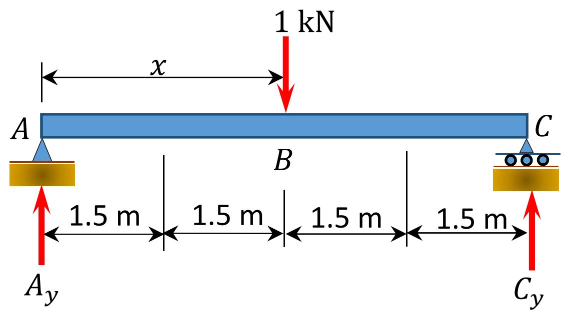
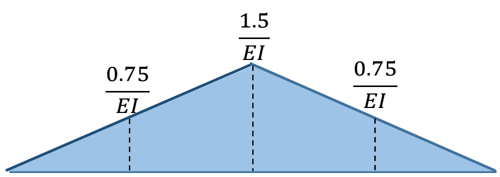


Fig. 13.3. Indeterminate beam.

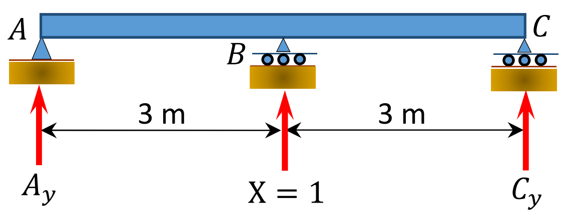


(b) Primary beam subjected to unit load at B

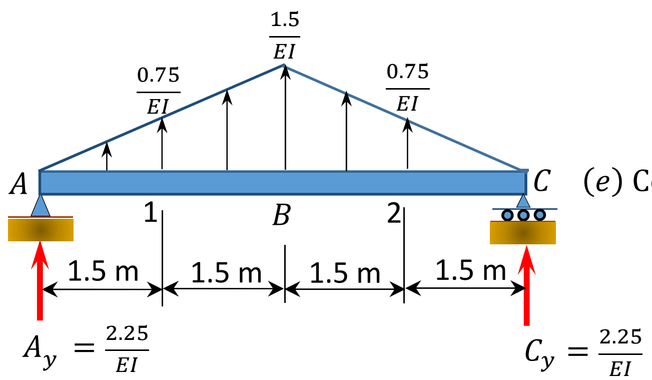
Solution



(c) $\frac{M}{EI}$- - diagram for primary beam subjected to unit load at B



(d) Primary beam subjected to redundant X = 1



(e) Conjugate beam for unit load at B

$$\delta_{\mathrm{BA}} = \delta_{\mathrm{BC}} = 0$$

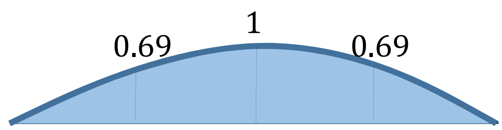
$$\delta_{\mathrm{B1}} = \delta_{\mathrm{B2}} = \left(\frac{2.25}{EI}\right)(1.5) - \left(\frac{1}{2}\right)(1.5)\left(\frac{0.75}{EI}\right)\left(\frac{1.5}{3}\right) = 3.09$$

$$\delta_{BB} = \left(\frac{2.25}{EI}\right)(3) - \left(\frac{1}{2}\right)(3)\left(\frac{1.5}{EI}\right)\left(\frac{3}{3}\right) = 4.50$$

$$\delta_{BX} = -\delta_{BB} = -4.5$$

When the unit load is at different points along the beam, the ordinate of the influence line for the redundant at B_y can be computed using the compatibility equation:

$B_y = -\frac{\delta_{BX}}{\delta_{BB}}$



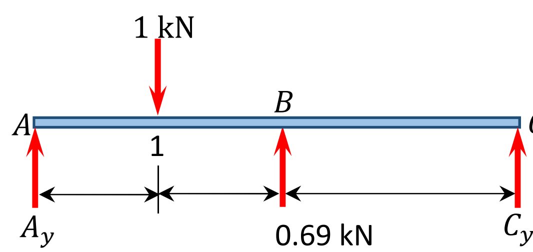
(f) Influence line for B_y

At point A and C, $B_y = 0$

At point 1 and 2, $B_y = \frac{3.09}{4.5} = 0.69$

At point B, $B_y = \frac{4.5}{4.5} = 1.0$

Now that B_y is known, the values of the ordinate of the influence lines for other reactions can be obtained using statics. For instance, to determine the ordinate of the influence line at point 1, place the unit load at point 1 and the value of the redundant when the unit load is at point 1 and solve as follows:



(g)

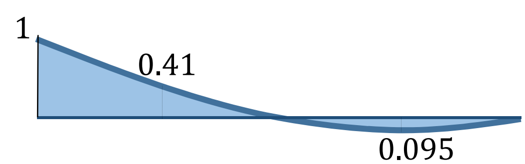
Ordinates of influence line for A_y.

$+\circlearrowleft \sum M_C = 0$

When the unit load is at point 1,

$-A_y(6) + 1(4.5) - 0.69(3) = 0$

$A_y = \frac{2.43}{6} = 0.41$



(h) Influence line for A_y

When the unit load is at point 2,

$-A_y(6) + 1(1.5) - 0.69(3) = 0$

$A_y = \frac{2.43}{6} = -\frac{0.57}{6} = -0.095$

When the unit load is at point A, $A_y = 1$

When the unit load is at point B and C, $A_y = 0$

Ordinates of Influence line for C_y.

$+\curvearrowleft \sum M_A = 0$

When the unit load is at point 1,

$C_y(6) - 1(1.5) + 0.69(3) = 0$

$C_y = -\frac{0.57}{6} = -0.095$

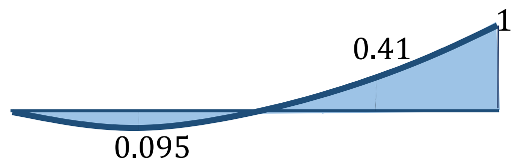
When the unit load is at point 2,

$C_y(6) - 1(4.5) + 0.69(3) = 0$

$A_y = \frac{2.43}{6} = 0.41$

When the unit load is at point C, $C_y = 1$

When the unit load is at point A and B, $C_y = 0$



(i) Influence line for B_y

13.3 Influence Lines for Statically Indeterminate Beams by Kinematic Method

In 1886, Heinrich Muller-Breslau, a German Professor, developed a procedure for the establishment of the shape of the influence lines for functions such as reactions, shears, moments, and axial forces in members without any computational effort. The influence lines obtained by this method are also referred to as qualitative influence lines, as there is no calculation involved. The Muller-Breslau method is based on the principle of virtual work. The procedure for this method, which is commonly referred to as Muller-Breslau's principle, is stated as follows:

> The influence line for any function such as a reaction, shear, or moment of a structure can be represented by the deflected shape of a release structure obtained by removing from the given structure the restraint that corresponds to the particular function being considered, and then introducing a unit displacement or rotation in the direction and the location of the function being considered.

When there is a need to obtain the ordinates for the influence lines while using the kinematic method, this procedure must be complemented by other analytical techniques, such as the method of singularity function, the Hardy Cross method of moment distribution, the energy methods, and the conjugate beam principle. In such instances, the procedure is as follows:

Procedure for Analysis of Influence Lines by the Kinematic Method

- Obtain the released structure by removing the restraint that corresponds to the function whose influence line is desired.
- Apply a unit displacement or rotation to the released structure in the direction and at the location of the function whose influence line is desired.
- Draw the deflected shape of the released structure. This corresponds to the influence line of the function being considered.
- Place a unit load at the location and in the direction of the function being considered, and find the value of the ordinate of the influence line using statics.
- Using geometry, determine the value of other ordinates of influence using geometry.

Example 13.3

Using the Muller-Breslau's principle, draw the qualitative influence lines for the vertical reactions at supports A, B, and C, the shear and bending moment at section X_1, and the bending moment at support D of the five-span beam shown in Figure 13.4a.

Solution

Qualitative influence line for the vertical reactions at support A, B, and C.

To draw the qualitative influence line for A_y, first obtain the release structure by removing the support at A. Applying a unit displacement at point A in the release structure, in the positive direction of A_y, will result in the deflected shape shown in Figure 13.4b. The resulting deflected shape represents the shape of the influence line of A_y. To obtain the shape of the influence lines for B_y and C_y, similar procedures are followed and will yield the deflected shapes shown in Figure13.4c and Figure13.4d.

Qualitative influence lines for the shear at section X_1.

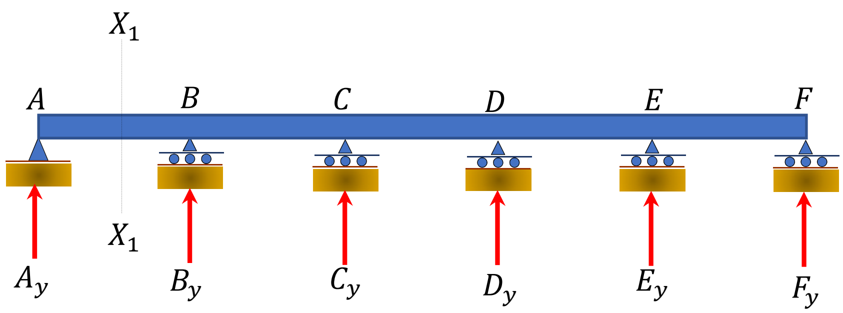
The qualitative influence line for the shear at section X_1 is drawn by first breaking the beam at the section and then applying two vertical forces in a manner that will cause a positive shear on the left and the right portion of the break. The resulting deflected shape is shown in Figure 13.4e.

Qualitative influence lines for the bending moment at section X_1.

The influence line of the moment at section X_1 is found by first inserting an imaginary hinge at the section X_1 and then applying a pair of positive bending moments adjacent to both sides of the hinge. The resulting deflected shape shown in Figure 13.4f represents the shape of the qualitative influence line for the bending moment at the section.

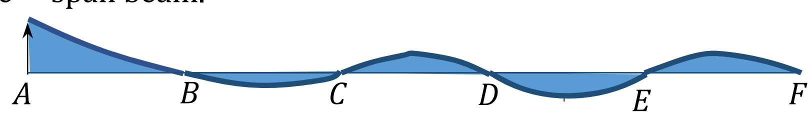
Qualitative influence lines for the bending moment at support D.

The influence line for the moment at the support D is obtained by first releasing the restrain at the support, inserting a pin at point D of the release structure, and then applying a pair of moments adjacent to both sides of the hinge in the positive direction of M_D. The resulting deflected shape shown in Figure 13.4g represents the shape of the qualitative influence line for the bending moment at the section.

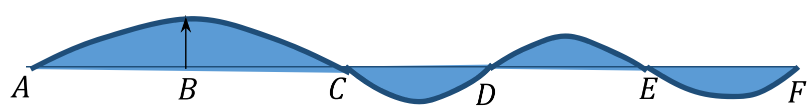


(a)

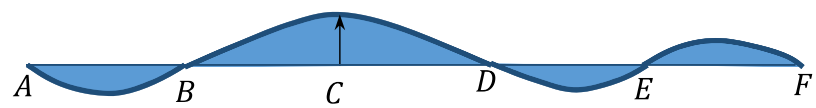
Fig. 13.4. Five − span beam.



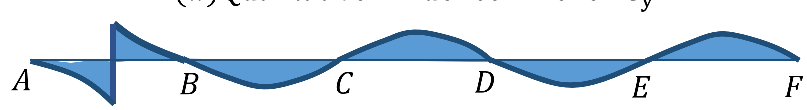
(b)Qualitative Influence Line for A_y



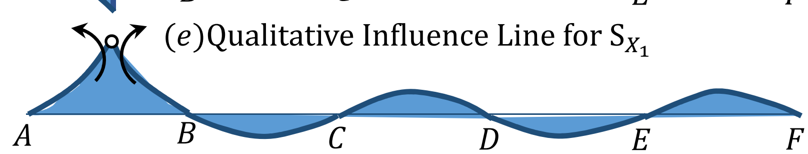
(c)Qualitative Influence Line for B_y



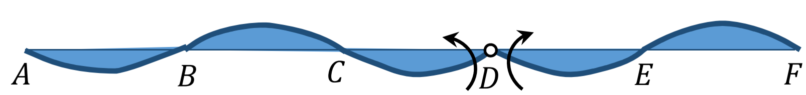
(d)Qualitative Influence Line for C_y



(e)Qualitative Influence Line for S_{X_1}



(f)Qualitative Influence Line for M_{X_1}



(g)Qualitative Influence Line for M_D

Chapter Summary

Influence lines for indeterminate structures: The procedure for the construction of influence lines for indeterminate structures by the equilibrium method and the Muller-Breslau principle were discussed, and a few example problems were solved in this chapter. Unlike the influence lines for determinate structures, which are straight lines, the influence line for indeterminate structures are curvilinear.

Practice Problems

13.1 Using the equilibrium method, draw the influence lines for the vertical reactions at *ACD* of the beam shown in Figure P13.1. Also, draw the influence line for the shear force and bending moment at a section at *B* of the beam.

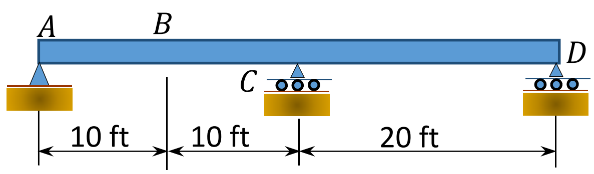


Fig. P 13.1. Beam. EI = constant

13.2 Using the equilibrium method, draw the influence lines for the vertical reactions at the supports of the indeterminate beam with overhanging ends, as shown in Figure P13.2.

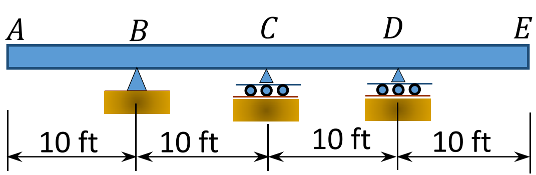


Fig. P 13.2. Indeterminate beam. EI = constant

13.3 Using the equilibrium method, draw the influence lines for the vertical reactions at supports *A* and *C* of the propped cantilever beam shown in Figure P13.3.

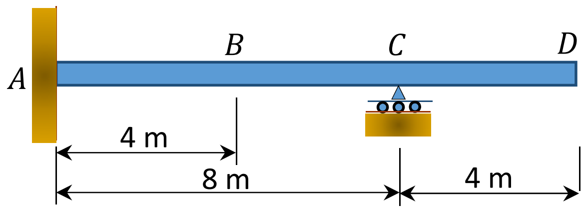


Fig. P 13.3. Propped cantilever beam. $EI = \text{constant}$

13.4 Using Muller-Breslau's principle, draw the qualitative influence lines for the vertical reactions at supports *A*, *B*, and *C*, positive shear and moment at section X_1.

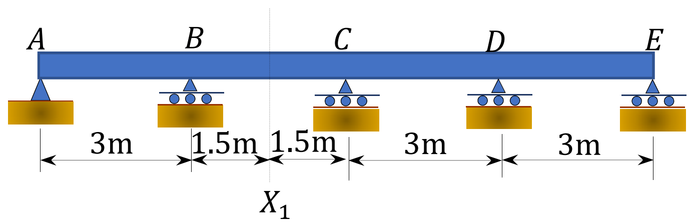


Fig. P13.4. Beam.

13.5 Using Muller-Breslau's principle, draw the qualitative influence lines for the vertical reactions at supports *E* and *F*, the negative moment at *C*, negative shear and moment at section X_1.

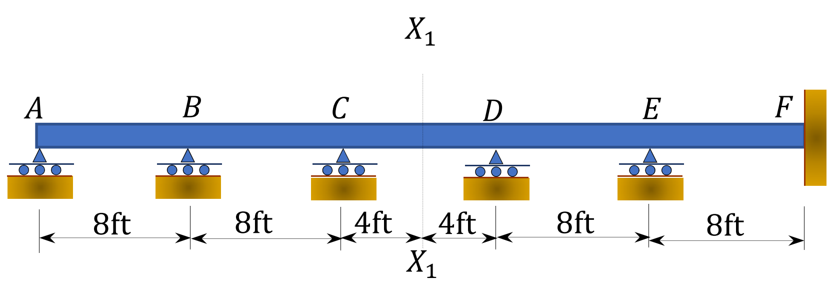


Fig. P13.5 Beam.

13.6 Using Muller-Breslau's principle, draw the qualitative influence lines for the maximum vertical reactions at supports *A* and *B*, maximum negative shear and moment at section X_1.

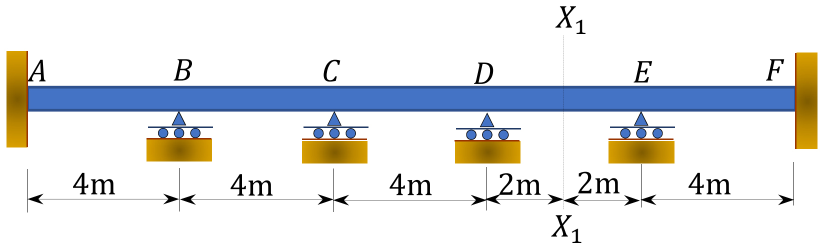


Fig. P13.6. Beam.